

Tannaka–Kreĭn reconstruction and a characterization of modular tensor categories

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January 30, 2008

Abstract

We show that every modular category is equivalent as an additive ribbon category to the category of finite-dimensional comodules of a Weak Hopf Algebra. This Weak Hopf Algebra is finite-dimensional, split cosemisimple, weakly cofactorizable, coribbon and has trivially intersecting base algebras. In order to arrive at this characterization of modular categories, we develop a generalization of Tannaka–Kreĭn reconstruction to the long version of the canonical forgetful functor which is lax and oplax monoidal, but not in general strong monoidal, thereby avoiding all the difficulties related to non-integral Frobenius–Perron dimensions. In the more general case of a finitely semisimple additive ribbon category, not necessarily modular, the reconstructed Weak Hopf Algebra is finite-dimensional, split cosemisimple, coribbon and has trivially intersecting base algebras.

Mathematics Subject Classification (2000): 16W30, 18D10

keywords: Modular category, tensor category, Weak Hopf Algebra, Tannaka–Kreĭn reconstruction

1 Introduction

A modular category [1, 2] is a finitely semisimple additive ribbon category that satisfies a non-degeneracy condition. For the precise definition, see Section 2.1 below. Modular categories are of interest in a variety of areas from low-dimensional topology [1, 2] to Conformal Field Theory [3, 4], subfactor theory [5] and 3-dimensional quantum gravity [6].

It is well known that some modular categories are equivalent to categories of the form ${}_A\mathcal{M}$, the category of left A -modules of some suitable ring or algebra A . Since algebras are often easier to deal with than categories, it is an interesting problem to understand whether all modular categories are of this form. We show that this is indeed the case. We restrict ourselves to modular categories \mathcal{C} for which the commutative ring $k = \text{End}(\mathbb{1})$, *i.e.* the endomorphisms of the monoidal unit object, is a field.

In many cases, it is outright obvious that a modular category \mathcal{C} is equivalent to the category ${}_A\mathcal{M}$ for some k -algebra A . This is the case, for example, if all simple objects V_j of \mathcal{C} are finite-dimensional vector spaces over some field k . Since \mathcal{C} is by definition finitely semisimple¹, A is just a finite direct sum of the appropriate $n_j \times n_j$ -matrix algebras with coefficients in k

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¹The assumption of finite semisimplicity includes that $\text{End}(V_j) \cong k$ for each simple object V_j of \mathcal{C} .

where $n_j = \dim_k V_j$. The equivalence $\mathcal{C} \simeq {}_A\mathcal{M}$ is just an equivalence of (ordinary) categories, and one still needs to determine which additional structure and properties of A give rise to the monoidal structure, braiding, ribbon structure and special properties of \mathcal{C} .

Which sort of additional structure on a k -algebra A would be sufficient in order to equip the category ${}_A\mathcal{M}$ of left A -modules with the structure of a monoidal category? The most widely known answer to this question is that one can employ the structure of a Hopf algebra or a bialgebra. It is further known, for example, that ribbon Hopf algebras H [7], a special sort of quasitriangular Hopf algebras, have categories ${}_H\mathcal{M}$ of left H -modules that carry the structure of ribbon categories. In order to answer the converse question, *i.e.* which ribbon categories are of the form ${}_H\mathcal{M}$ for some k -algebra H , Tannaka–Kreĭn reconstruction [8, 9] was generalized from (the coordinate rings of) groups to Hopf algebras, ribbon Hopf algebras and even quasi Hopf algebras, see, for example [10–14]. These constructions successfully deal with the additional structure such as duality, braiding and the ribbon structure, but quite a basic problem with the monoidal structure is left unsolved.

The problem is that not every rigid monoidal category is monoidally equivalent to the category of modules over a Hopf algebra or a quasi Hopf algebra. For example, a modular category \mathcal{C} with $\text{End}(\mathbb{1}) = \mathbb{C}$ is the category of H -modules for some finite-dimensional quasi Hopf algebra H if and only if each simple object of \mathcal{C} has an integer Frobenius–Perron dimension [15, Theorem 8.33]. But there exist interesting examples of modular categories that contain objects of non-integer Frobenius–Perron dimension².

In order to deal with non-integer Frobenius–Perron dimensions, Böhm, Nill and Szlachányi have invented the concept of Weak Bialgebras (WBAs) and Weak Hopf Algebras (WHAs) [17–22]. Böhm’s thesis [23] contains the first examples of modular categories which have objects of non-integer Frobenius–Perron dimension and which are shown to be the categories of modules of some finite-dimensional WBA. The definitions of a WBA and of a WHA are summarized in detail in Section 2.2 below.

Is the concept of a WBA general enough in order to show that every modular category \mathcal{C} is equivalent (first as a monoidal and then as a ribbon category) to the category of modules of some WBA H ? It is useful to subdivide this question into the following three steps:

- (1) Can every object $X \in |\mathcal{C}|$ be viewed as a k -vector space for some k ?
- (2) Does the monoidal structure of \mathcal{C} arise from the WBA structure of H ?
- (3) Which additional structure and properties of H are required in order to obtain duality, braiding and ribbon structure of \mathcal{C} and in order to satisfy the non-degeneracy condition?

Question (1) was answered by Hayashi [24] who showed that there is a canonical forgetful functor $\widehat{\omega}: \mathcal{C} \rightarrow {}_R\mathcal{M}_R$ into the category ${}_R\mathcal{M}_R$ of (R, R) -bimodules. Here $R = \text{End}(\widehat{V})$ is the commutative k -algebra, $k = \text{End}(\mathbb{1})$, of endomorphisms of the universal object,

$$\widehat{V} = \bigoplus_{j \in I} V_j, \quad (1.1)$$

the direct sum over one representative V_j for each isomorphism class of simple objects.

²Important examples of modular categories are related to (a finite-dimensional version of) the Hopf algebras $U_q(\mathfrak{g})$, q a root of unity, see, for example [1, 16]. These are, however, not ${}_{U_q(\mathfrak{g})}\mathcal{M}$. One rather has to take first the full subcategory of tilting modules of ${}_{U_q(\mathfrak{g})}\mathcal{M}$ and then a quotient of that subcategory, dividing out the negligible morphisms. The resulting modular category is in general no longer of the form ${}_H\mathcal{M}$ for any Hopf algebra H .

This functor $\widehat{\omega}: \mathcal{C} \rightarrow {}_R\mathcal{M}_R$ is now known as the *short forgetful functor*. In order to solve question (1) above, one composes it with the forgetful functor ${}_R\mathcal{M}_R \rightarrow \mathbf{Vect}_k$ that assigns to each (R, R) -bimodule the underlying k -vector space, and thereby obtains the *long forgetful functor* $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$. While the short forgetful functor is strong monoidal, the long forgetful functor is in general not strong monoidal and therefore not a *fibre functor* in the usual technical sense. Szlachányi [25] has characterized those long forgetful functors that originate from the categories of modules of WBAs.

Hayashi [24] and Háí [26] have studied the generalization of Tannaka–Kreĭn reconstruction to the case of the short forgetful functor, *i.e.* to a strong monoidal functor into the bimodule category ${}_R\mathcal{M}_R$. It is known that the reconstructed algebraic structure is a bialgebroid over R and, furthermore, since R is a finite-dimensional separable commutative k -algebra, one actually gets a WBA [25]. Therefore, Ostrik [27] concludes from these abstract considerations that the answer to question (2) above is ‘yes’.

Tannaka–Kreĭn reconstruction using the short forgetful functor $\widehat{\omega}: \mathcal{C} \rightarrow {}_R\mathcal{M}_R$ alone, however, uses the language of bialgebroids, and it is thus not transparent how duality, braiding and ribbon structure carry over from the modular category to the reconstructed WBA.

It is the purpose of the present article to complete the programme of Tannaka–Kreĭn reconstruction including question (3) above, and to prove the following

Theorem 1.1. Every modular category for which $k = \text{End}(\mathbb{1})$ is a field, is equivalent as a k -linear ribbon category to the category of finite-dimensional comodules of a finite-dimensional split cosemisimple weakly cofactorizable coribbon WHA over k whose base algebras intersect trivially.

This theorem also holds, more generally, without the non-degeneracy condition on the S -matrix:

Theorem 1.2. Every finitely semisimple additive ribbon category for which $k = \text{End}(\mathbb{1})$ is a field, is equivalent as a k -linear ribbon category to the category of finite-dimensional comodules of a finite-dimensional split cosemisimple coribbon WHA over k whose base algebras intersect trivially.

We reconstruct this WHA, characterize all its operations by the universal property of the appropriate coend, *i.e.* the universal coacting coalgebra, and also write down the operations in terms of a convenient basis.

Several authors have given sufficient conditions for the category of modules ${}_A\mathcal{M}$ of some k -algebra A to be modular, see, for example [28, Lemma 1.1] for Drinfel’d doubles of Hopf algebras and [29, Lemma 8.2] for WHAs. As far as we know, Theorem 1.1 is the first one to establish the precise form of the converse implication, *i.e.* that every modular category can indeed be obtained from a WHA with the properties stated.

In order to prove Theorem 1.1, we generalize Tannaka–Kreĭn reconstruction to the long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$. Since this functor has the category \mathbf{Vect}_k of vector spaces over k as its codomain, reconstruction immediately yields a coalgebra object³ H in \mathbf{Vect}_k . This is substantially more transparent than a functor into the bimodule category and allows us to recover all additional operations of H by exploiting the universal property of the coend.

³For the usual technical reasons, *i.e.* because we want to exploit that the category \mathbf{Vect}_k is (small) cocomplete and its tensor product preserves colimits in both arguments, we prefer to reconstruct a coalgebra rather than an algebra. For more details, we refer to Section 2.4.

We emphasize that the long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ is in general not strong monoidal, but nevertheless both lax and oplax monoidal [25], and we have to generalize Tannaka–Kreĭn reconstruction to this case. It is this property of being lax and oplax rather than strong monoidal that enables us to deal with non-integer Frobenius–Perron dimensions.

The present article is structured as follows. In Section 2, we review the definitions and some key results on modular categories, WHAs, comodules, and on Tannaka–Kreĭn reconstruction. In Section 3, we study the properties of the long forgetful functor. We reconstruct a coribbon WHA from each modular category in Section 4. In Section 5, we study the category of finite-dimensional comodules of a coribbon WHA, and in Section 6, we show that the original modular category is equivalent to the category of finite-dimensional comodules of the reconstructed coribbon WHA. For convenient reference, we compile the relevant definitions and results about monoidal categories in Appendix A.

The reader who just wants to get a quick overview of how the reconstructed WHA looks like, without going through all the technical details, is invited to go straight to Section 7 where we present the reconstructed WHA for the modular category associated with $U_q(\mathfrak{sl}_2)$, q a root of unity, in term of the familiar diagrams.

2 Preliminaries

2.1 Modular categories

In this section, we summarize the definition and some basic properties of modular categories. For more details, we refer to the book [2].

Our notation is as follows. If \mathcal{C} is a category, we write $X \in |\mathcal{C}|$ for the objects X of \mathcal{C} , $\mathrm{Hom}(X, Y)$ for the collection of all morphisms $f: X \rightarrow Y$ and $\mathrm{End}(X) = \mathrm{Hom}(X, X)$. By $\mathrm{id}_X: X \rightarrow X$ we denote the identity morphism of X and by $g \circ f: X \rightarrow Z$ the composition of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$. If two objects $X, Y \in |\mathcal{C}|$ are isomorphic, we write $X \cong Y$. If two categories are equivalent, we write $\mathcal{C} \simeq \mathcal{D}$. The identity functor on \mathcal{C} is denoted by $1_{\mathcal{C}}$, and $\mathcal{C}^{\mathrm{op}}$ is the opposite category of \mathcal{C} . The category of vector spaces over a field k is denoted by \mathbf{Vect}_k and its full subcategory of finite-dimensional vector spaces by \mathbf{fdVect}_k .

We assume that the reader is familiar with the notions of **Ab**-enriched, additive, abelian, monoidal, braided monoidal, autonomous and ribbon categories. For convenience, we have compiled the relevant definitions in Appendix A.

Definition 2.1. A *modular category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \mathrm{ev}, \mathrm{coev}, \sigma, \nu)$ is an additive ribbon category (c.f. Definitions A.11 and A.18) that satisfies the following conditions:

- (1) $k = \mathrm{End}(\mathbb{1})$ is a field.
- (2) There is a finite family $\{V_j\}_{j \in I}$ of objects $V_j \in |\mathcal{C}|$ where I denotes some finite index set, that satisfies the following conditions:
 - (a) Each V_j , $j \in I$, satisfies $\mathrm{End}(V_j) \cong k$, i.e. it is *simple*.
 - (b) There is an element $0 \in I$ such that $V_0 \cong \mathbb{1}$.
 - (c) For each $j \in I$, there is some $j^* \in I$ such that $V_{j^*} \cong (V_j)^*$.
 - (d) For each object $X \in |\mathcal{C}|$, there is a finite sequence $(j_1^X, \dots, j_{n^X}^X) \in I^{n^X}$, $n^X \in \mathbb{N}_0$,

and morphisms $\iota_\ell^X : V_{j_\ell^X} \rightarrow X$ and $\pi_\ell^X : X \rightarrow V_{j_\ell^X}$ for $1 \leq \ell \leq n$ such that

$$\text{id}_X = \sum_{\ell=1}^{n^X} \iota_\ell^X \circ \pi_\ell^X. \quad (2.1)$$

(3) The matrix $(S_{ij})_{i,j \in I}$ (*S-matrix*) whose coefficients are

$$S_{ij} = \text{tr}_{V_i \otimes V_j}(\sigma_{V_j, V_i} \circ \sigma_{V_i, V_j}) \in k, \quad (2.2)$$

is invertible.

Compared with the definition of Turaev [2], we have added in our Definition 2.1 the conditions that $k = \text{End}(\mathbb{1})$ be a field and that \mathcal{C} be additive rather than just **Ab**-enriched. The former is related to the fact that we reconstruct a WHA over k and we only deal with the case in which this is a field. The latter makes sure that \mathcal{C} has all finite biproducts ('direct sums'). Otherwise, one could remove some of the objects of \mathcal{C} that are biproducts of simple objects, without violating any condition of the definition. We disallow this because we want to compare \mathcal{C} to the category of comodules of the reconstructed WHA which automatically has all finite biproducts.

Note that in the definition of a modular category, one usually requires $\text{End}(V_j) \cong k$ for the simple objects although one does not impose any restriction on the field k such as algebraic closure. Many algebraic examples of modular categories, see, for example [16], have $k = \mathbb{Q}(\varepsilon)$, a cyclotomic extension of the rationals, far from algebraically closed, and nevertheless $\text{End}(V_j) \cong k$ for all simple objects.

Proposition 2.2. Let \mathcal{C} be a modular category, $k = \text{End}(\mathbb{1})$, and $\{V_j\}_{j \in I}$ be a family of objects as in Definition 2.1(2).

- (1) \mathcal{C} is k -linear as a monoidal category (c.f. Definition A.18) [2, Section I.1.5].
- (2) For all objects $X, Y \in |\mathcal{C}|$, the abelian group $\text{Hom}(X, Y)$ is a finite-dimensional vector space over k [2, Lemma II.4.2.1].
- (3) \mathcal{C} is non-degenerate, i.e. its traces define non-degenerate bilinear forms (c.f. Definition A.25) [2, Lemma II.4.2.3].
- (4) The morphisms ι_ℓ^X and π_ℓ^X of Definition 2.1(2d) can be chosen in such a way that

$$\pi_\ell^X \circ \iota_m^X = \begin{cases} \text{id}_{V_{j_\ell^X}}, & \text{if } \ell = m, \\ 0, & \text{else} \end{cases} \quad (2.3)$$

for all $\ell, m \in I$ (Proposition A.29).

- (5) If $j, \ell \in I$ and $j \neq \ell$, then $\text{Hom}(V_j, V_\ell) = \{0\}$ [2, Lemma II.1.5].
- (6) \mathcal{C} is finitely semisimple according to Definition A.20(3).
- (7) If $X \in |\mathcal{C}|$ is simple, then $0 \neq \dim(X) \in k$ [2, Lemma II.4.2.4].
- (8) If $X \in |\mathcal{C}|$ is simple, then there exists some $j \in I$ such that $X \cong V_j$ (Corollary A.23).

Most results of this article already hold without the non-degeneracy condition on the *S*-matrix (2.2), i.e. for finitely semisimple additive ribbon categories for which $k = \text{End}(\mathbb{1})$ is a field.

2.2 Weak Hopf algebras

In this section, we summarize the definitions of a Weak Bialgebra (WBA) and of a Weak Hopf Algebra (WHA). For more details, we refer to [18–22].

Definition 2.3. A *Weak Bialgebra* $(H, \mu, \eta, \Delta, \varepsilon)$ over a field k is a k -vector space H with linear maps $\mu: H \otimes H \rightarrow H$ (*multiplication*), $\eta: k \rightarrow H$ (*unit*), $\Delta: H \rightarrow H \otimes H$ (*comultiplication*), and $\varepsilon: H \rightarrow k$ (*counit*) such that the following conditions hold:

- (1) (H, μ, η) is an associative unital algebra, i.e. $\mu \circ (\mu \otimes \text{id}_H) = \mu \circ (\text{id}_H \otimes \mu)$ and $\mu \circ (\eta \otimes \text{id}_H) = \text{id}_H = \mu \circ (\text{id}_H \otimes \eta)$.
- (2) (H, Δ, ε) is a coassociative counital coalgebra, i.e. $(\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes \text{id}_H) \circ \Delta = \text{id}_H = (\text{id}_H \otimes \varepsilon) \circ \Delta$.
- (3) The following compatibility conditions hold:

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (\text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H) \circ (\Delta \otimes \Delta), \quad (2.4)$$

$$\begin{aligned} \varepsilon \circ \mu \circ (\mu \otimes \text{id}_H) &= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\text{id}_H \otimes \Delta \otimes \text{id}_H) \\ &= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\text{id}_H \otimes \Delta^{\text{op}} \otimes \text{id}_H), \end{aligned} \quad (2.5)$$

$$\begin{aligned} (\Delta \otimes \text{id}_H) \circ \Delta \circ \eta &= (\text{id}_H \otimes \mu \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta) \\ &= (\text{id}_H \otimes \mu^{\text{op}} \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta). \end{aligned} \quad (2.6)$$

Here $\sigma_{V,W}: V \otimes W \rightarrow W \otimes V$, $v \otimes w \mapsto w \otimes v$ is the transposition of the tensor factors in \mathbf{Vect}_k , and by $\Delta^{\text{op}} = \sigma_{H,H} \circ \Delta$ and $\mu^{\text{op}} = \mu \circ \sigma_{H,H}$, we denote the *opposite comultiplication* and *opposite multiplication*, respectively. We tacitly identify the vector spaces $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$ and $V \otimes k \cong V \cong k \otimes V$, exploiting the coherence theorem for the monoidal category \mathbf{Vect}_k .

We use the term *comultiplication* for the operation Δ in a coalgebra whereas *coproduct* always refers to a colimit in a category.

Definition 2.4. A *homomorphism* $\varphi: H \rightarrow H'$ of WBAs $(H, \mu, \eta, \Delta, \varepsilon)$ and $(H', \mu', \Delta', \varepsilon')$ over the same field k is a k -linear map that is a homomorphism of unital algebras, i.e. $\varphi \circ \eta = \eta'$ and $\varphi \circ \mu = \mu' \circ (\varphi \otimes \varphi)$, as well as a homomorphism of counital coalgebras, i.e. $\varepsilon' \circ \varphi = \varepsilon$ and $\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$.

Definition 2.5. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a WBA. The linear maps $\varepsilon_t: H \rightarrow H$ (*target counital map*) and $\varepsilon_s: H \rightarrow H$ (*source counital map*) are defined by

$$\varepsilon_t := (\varepsilon \otimes \text{id}_H) \circ (\mu \otimes \text{id}_H) \circ (\text{id}_H \otimes \sigma_{H,H}) \circ (\Delta \otimes \text{id}_H) \circ (\eta \otimes \text{id}_H), \quad (2.7)$$

$$\varepsilon_s := (\text{id}_H \otimes \varepsilon) \circ (\text{id}_H \otimes \mu) \circ (\sigma_{H,H} \otimes \text{id}_H) \circ (\text{id}_H \otimes \Delta) \circ (\text{id}_H \otimes \eta). \quad (2.8)$$

Both ε_t and ε_s are idempotents. A WBA $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra if and only if $\Delta \circ \eta = \eta \otimes \eta$, if and only if $\varepsilon \circ \mu = \varepsilon \otimes \varepsilon$, if and only if $\varepsilon_s = \eta \circ \varepsilon$ and if and only if $\varepsilon_t = \eta \circ \varepsilon$.

Proposition 2.6. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a WBA.

- (1) The subspace $H_t := \varepsilon_t(H)$ (*target base algebra*) forms a subalgebra with unit and a left coideal, i.e.

$$\Delta(H_t) \subseteq H \otimes H_t. \quad (2.9)$$

- (2) The subspace $H_s := \varepsilon_s(H)$ (*source base algebra*) forms a subalgebra with unit and a right coideal, i.e.

$$\Delta(H_s) \subseteq H_s \otimes H. \quad (2.10)$$

- (3) The subalgebras H_s and H_t commute, i.e. $xy = yx$ for all $x \in H_t$ and $y \in H_s$.

Definition 2.7. A *Weak Hopf Algebra* $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a Weak Bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ with a linear map $S: H \rightarrow H$ (*antipode*) that satisfies the following conditions:

$$\mu \circ (\text{id}_H \otimes S) \circ \Delta = \varepsilon_t, \quad (2.11)$$

$$\mu \circ (S \otimes \text{id}_H) \circ \Delta = \varepsilon_s, \quad (2.12)$$

$$\mu \circ (\mu \otimes \text{id}_H) \circ (S \otimes \text{id}_H \otimes S) \circ (\Delta \otimes \text{id}_H) \circ \Delta = S. \quad (2.13)$$

For convenience, we write $1 = \eta(1)$ and omit parentheses in products, exploiting associativity. We also use Sweedler's notation and write $\Delta(x) = x' \otimes x''$ for the comultiplication of $x \in H$ as an abbreviation of the expression $\Delta(x) = \sum_k a_k \otimes b_k$ with some $a_k, b_k \in H$. Similarly, we write $((\Delta \otimes \text{id}_H) \circ \Delta)(x) = x' \otimes x'' \otimes x'''$, exploiting coassociativity. Then, for example, equation (2.7) reads $\varepsilon_t(x) = \varepsilon(1'x)1''$ for all $x \in H$.

The concepts of a WBA and of a WHA are formally self-dual, i.e. if H is a [WBA, WHA] that is finite-dimensional as a vector space, then its dual space H^* is a [WBA, WHA] as well.

Definition 2.8. A *homomorphism* $\varphi: H \rightarrow H'$ of WHAs is a homomorphism of WBAs for which $\varphi \circ S = S' \circ \varphi$.

If H is a WHA, we denote by H^{op} its opposite WHA $(H, \mu^{\text{op}}, \eta, \Delta, \varepsilon, S^{-1})$, by H^{cop} its coopposite WHA $(H, \mu, \eta, \Delta^{\text{op}}, \varepsilon, S^{-1})$ and by $H^{\text{op}, \text{cop}}$ the WHA $(H, \mu^{\text{op}}, \eta, \Delta^{\text{op}}, \varepsilon, S)$. The antipode of a WHA is an algebra antihomomorphism, i.e. $S \circ \mu = \mu^{\text{op}} \circ (S \otimes S)$ and $S \circ \eta = \eta$, as well as a coalgebra antihomomorphism, i.e. $(S \otimes S) \circ \Delta = \Delta^{\text{op}} \circ S$ and $\varepsilon \circ S = \varepsilon$.

Definition 2.9 (see [21, 22]). Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a WHA.

- (1) The *minimal Weak Hopf Algebra* H_{\min} of H is the smallest sub WHA of H that contains the unit $\eta(1) \in H$.
- (2) H is called *regular* if $S^2|_{H_{\min}} = \text{id}_{H_{\min}}$.

2.3 Coalgebras and comodules

The following definitions and results can be found, for example, in [30].

Definition 2.10. Let (C, Δ, ε) be a coalgebra over some field k . A right C -comodule (V, β_V) is a k -vector space V with a k -linear map $\beta_V: V \rightarrow V \otimes C$ that satisfies

$$(\text{id}_V \otimes \varepsilon) \circ \beta_V = \text{id}_V, \quad (2.14)$$

$$(\beta_V \otimes \text{id}_C) \circ \beta_V = (\text{id}_V \otimes \Delta) \circ \beta_V. \quad (2.15)$$

Definition 2.11. Let (C, Δ, ε) be a coalgebra and (V, β_V) and (W, β_W) be right C -comodules. A *morphism of coalgebras* $f: V \rightarrow W$ is a k -linear map that satisfies

$$(f \otimes \text{id}_C) \circ \beta_V = \beta_W \circ f. \quad (2.16)$$

We extend Sweedler's notation to comodules and write $\beta(v) = v_V \otimes v_C$. The conditions (2.14) and (2.15) then read $v_V \varepsilon(v_C) = v$ and $(v_V)_V \otimes (v_V)_C \otimes v_C = v_V \otimes (v_C)' \otimes (v_C)''$.

Proposition 2.12. Let (C, Δ, ε) be a coalgebra over some field k and \mathcal{M}^C be the category whose objects are the right C -comodules that are finite-dimensional as vector spaces over k , and whose morphisms are morphisms of right C -comodules. Then \mathcal{M}^C is k -linear and abelian, and $\text{Hom}(V, W)$ is finite-dimensional for all $V, W \in |\mathcal{M}^C|$.

If V is a finite-dimensional right C -comodule with basis $(v_j)_j$, then there are elements $c_{\ell j} \in C$ uniquely determined by the condition that $\beta_V(v_j) = \sum_{\ell} v_{\ell} \otimes c_{\ell j}$. They are called the *coefficients* of V with respect to that basis. They span the *coefficient coalgebra* $C(V) = \text{span}_k\{c_{\ell j}\}$, a sub coalgebra of C .

Let W be a finite-dimensional vector space over k with dual space W^* and a pair of dual bases $(e_j)_j$ and $(e^j)_j$ of W and W^* , respectively. We abbreviate $c_{jk} = e^j \otimes e_k \in W^* \otimes W$. The coalgebra $(W^* \otimes W, \Delta, \varepsilon)$ with $\Delta(c_{jk}) = \sum_{\ell} c_{j\ell} \otimes c_{\ell k}$ and $\varepsilon(c_{jk}) = \delta_{jk}$ is called the *matrix coalgebra* associated with W . In this case, W is a right $W^* \otimes W$ -comodule, and $W^* \otimes W$ is its coefficient coalgebra.

Definition 2.13. A coalgebra (C, Δ, ε) over a field k is called *cosimple* if C has no sub coalgebras other than C and $\{0\}$. The coalgebra C is called *cosemisimple* if it is a coproduct in \mathbf{Vect}_k of cosimple coalgebras. The coalgebra C is called *split cosemisimple* if it is cosemisimple and every cosimple sub coalgebra is a matrix coalgebra.

We prefer the term *cosemisimple* rather than the more common *semisimple* because it indicates that this is a property of a coalgebra. In the following, *semisimple WHA* therefore means that the underlying algebra of the WHA is semisimple whereas *cosemisimple WHA* means that its underlying coalgebra has the property just defined above.

Definition 2.14. Let (C, Δ, ε) be a coalgebra over a field k . A right C -comodule (V, β_V) is called *irreducible* if $V \neq \{0\}$ and V has no sub comodules other than V and $\{0\}$.

We here use the term *irreducible* as opposed to *simple* in order to distinguish it from the property that an object X of a k -linear category satisfies $\text{End}(X) \cong k$.

Lemma 2.15. Let (C, Δ, ε) be a coalgebra over a field k .

- (1) Every irreducible right C -comodule is finite-dimensional as a vector space over k .
- (2) If V and W are irreducible right C -comodules and $V \not\cong W$, then $\text{Hom}(V, W) = \{0\}$.
- (3) If C is split cosemisimple and V an irreducible right C -comodule, then $\text{End}(V) \cong k$.
- (4) If C is cosemisimple and V a finite-dimensional right C -comodule, then

$$V \cong \bigoplus_{i=1}^n V_i \tag{2.17}$$

for some irreducible right C -comodules V_i and $n \in \mathbb{N}_0$.

2.4 Tannaka–Kreĭn reconstruction

In this section, we summarize the main results on Tannaka–Kreĭn reconstruction of a coalgebra from a category \mathcal{C} with a functor $\mathcal{C} \rightarrow \mathbf{Vect}_k$, following [13].

Let \mathcal{C} be a small category and $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be a functor taking values in \mathbf{fdVect}_k . Then the *coend*

$$\mathbf{coend}(\mathcal{C}, \omega) = \int^{X \in |\mathcal{C}|} \omega(X)^* \otimes \omega(X) \quad (2.18)$$

exists.

In the following, we ignore all set theoretic issues and no longer mention the requirement that \mathcal{C} be small. In fact, all examples relevant to topology and mathematical physics that we are aware of, can already be obtained with essentially small \mathcal{C} , and whenever a coend appears, we can therefore replace \mathcal{C} by an equivalent small category.

By $\mathbf{Nat}(\omega, \omega \otimes -): \mathbf{Vect}_k \rightarrow \mathbf{Set}$ we denote the functor that sends each vector space M to the set $\mathbf{Nat}(\omega, \omega \otimes M)$ of natural transformations $\omega \Rightarrow \omega \otimes M$ and each linear map $\varphi: M \rightarrow N$ to the map of sets $(\mathrm{id}_\omega \otimes \varphi) \circ - : \mathbf{Nat}(\omega, \omega \otimes M) \rightarrow \mathbf{Nat}(\omega, \omega \otimes N)$.

Theorem 2.16. Let \mathcal{C} be a category and $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be a functor taking values in \mathbf{fdVect}_k . For a any vector space C , the following are equivalent:

- (1) $C \cong \mathbf{coend}(\mathcal{C}, \omega)$.
- (2) The functor $\mathbf{Nat}(\omega, \omega \otimes -): \mathbf{Vect}_k \rightarrow \mathbf{Set}$ is representable with representing object C .
- (3) There is a natural transformation $\delta^\omega: \omega \Rightarrow \omega \otimes C$ such that for each vector space M and each natural transformation $\varphi: \omega \Rightarrow \omega \otimes M$, there is a unique linear map $f: C \rightarrow M$ such that the diagram

$$\begin{array}{ccc} \omega & \xrightarrow{\delta^\omega} & \omega \otimes C \\ & \searrow \varphi & \downarrow \mathrm{id}_\omega \otimes f \\ & & \omega \otimes M \end{array} \quad (2.19)$$

of natural transformations between functors $\mathcal{C} \rightarrow \mathbf{Vect}_k$ commutes.

Proposition 2.17. Let \mathcal{C} be a category and $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be a functor taking values in \mathbf{fdVect}_k . The vector space $C = \mathbf{coend}(\mathcal{C}, \omega)$ forms a coassociative counital coalgebra (C, Δ, ε) . The operations $\Delta: C \rightarrow C \otimes C$ and $\varepsilon: C \rightarrow k$ are determined from the universal property of the coend by commutativity of the following diagrams of natural transformations between functors $\mathcal{C} \rightarrow \mathbf{Vect}_k$:

$$\begin{array}{ccc} \omega & \xrightarrow{d^\omega} & \omega \otimes C \\ \delta^\omega \downarrow & \searrow & \downarrow \mathrm{id}_\omega \otimes \Delta \\ \omega \otimes C & & \\ \delta^\omega \otimes \mathrm{id}_C \downarrow & & \downarrow \\ (\omega \otimes C) \otimes C & \xrightarrow{\alpha_{\omega(-), C, C}} & \omega \otimes (C \otimes C) \end{array} \quad (2.20)$$

and

$$\begin{array}{ccc}
 \omega & \xrightarrow{\delta^\omega} & \omega \otimes C \\
 & \searrow \rho_{\omega(-)}^{-1} & \downarrow \text{id}_\omega \otimes \varepsilon \\
 & & \omega \otimes k
 \end{array} \tag{2.21}$$

Here, α and ρ denote the associator and the right unit constraint of \mathbf{Vect}_k . We always draw the diagonal in these diagrams in order to remind the reader of (2.19).

Part (3) of Theorem 2.16 thus states that the coend $C = \mathbf{coend}(\mathcal{C}, \omega)$ is the universal coalgebra that coacts on all objects of \mathcal{C} . The coaction of C on the vector space $\omega(X)$ associated with an object $X \in |\mathcal{C}|$ is given by $\delta_X^\omega: \omega(X) \rightarrow \omega(X) \otimes C$. The following proposition describes $\mathbf{coend}(\mathcal{C}, \omega)$ as a vector space in terms of generators and relations.

Proposition 2.18. Let \mathcal{C} be a category and $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be a functor taking values in \mathbf{fdVect}_k . The coend is the vector space,

$$\mathbf{coend}(\mathcal{C}, \omega) \cong \left(\coprod_{X \in |\mathcal{C}|} \omega(X)^* \otimes \omega(X) \right) / N, \tag{2.22}$$

where \coprod denotes the coproduct in the category \mathbf{Vect}_k and

$$N = \{ (\omega(f)^* \vartheta) \otimes v - \vartheta \otimes (\omega(f)v) \mid \vartheta \in \omega(Y)^*; v \in \omega(X); f: X \rightarrow Y; X, Y \in |\mathcal{C}| \}. \tag{2.23}$$

The coalgebra structure of the coend is a quotient modulo N of a coproduct of matrix coalgebras. Let $(\omega(X)^*, \text{ev}_{\omega(X)}, \text{coev}_{\omega(X)})$ be a left-dual of $\omega(X)$ in \mathbf{Vect}_k . Such a left-dual exists because $\omega(X)$ is by assumption finite-dimensional. Then the structure of the coalgebra $\mathbf{coend}(\mathcal{C}, \omega)$ is given on the homogeneous elements of (2.22) by

$$\begin{aligned}
 \Delta: \omega(X)^* \otimes \omega(X) &\rightarrow (\omega(X)^* \otimes \omega(X)) \otimes (\omega(X)^* \otimes \omega(X)), \\
 \vartheta \otimes v &\mapsto \sum_j \vartheta \otimes e_j^{(X)} \otimes e_{(X)}^j \otimes v,
 \end{aligned} \tag{2.24}$$

$$\begin{aligned}
 \varepsilon: \omega(X)^* \otimes \omega(X) &\rightarrow k, \\
 \vartheta \otimes v &\mapsto \text{ev}_{\omega(X)}(\vartheta \otimes v).
 \end{aligned} \tag{2.25}$$

Here we have written $\text{coev}_{\omega(X)}(1) = \sum_j e_j^{(X)} \otimes e_{(X)}^j$. The universal coaction of $\mathbf{coend}(\mathcal{C}, \omega)$ on $\omega(X)$ is given by

$$\delta_X^\omega: \omega(X) \rightarrow \omega(X) \otimes (\omega(X)^* \otimes \omega(X)), \quad v \mapsto \sum_j e_j^{(X)} \otimes e_{(X)}^j \otimes v. \tag{2.26}$$

Below, we make use of this reconstruction of the coalgebra $\mathbf{coend}(\mathcal{C}, \omega)$ in the context in which \mathcal{C} is a modular category and ω the long forgetful functor.

In this section, we have used the fact that the category \mathbf{Vect}_k is small cocomplete and that the tensor product \otimes preserves colimits in both arguments.

3 The long forgetful functor

Let us now define the long forgetful functor by composing the canonical functor $\widehat{\omega}: \mathcal{C} \rightarrow {}_R\mathcal{M}_R$ of [24] with the forgetful functor ${}_R\mathcal{M}_R \rightarrow \mathbf{Vect}_k$ and show that this functor satisfies Szlachányi's conditions [25], i.e. that the functor is equipped with a *separable Frobenius structure*. Before we can show this, we need to establish some facts about modular categories and their non-degenerate traces. In this section, unless specified otherwise, $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$ is a finitely semisimple additive ribbon category for which $k = \text{End}(\mathbb{1})$ is a field. $\{V_j\}_{j \in I}$ denotes a family of objects as in Definition A.20(3).

3.1 Traces and convenient bases

The traces of the ribbon category \mathcal{C} can be used in order to relate $\text{Hom}(X, Y)^*$ with $\text{Hom}(Y, X)$, $X, Y \in |\mathcal{C}|$. For the reconstruction, it turns out to be convenient if one ‘rescales’ the traces by the following isomorphisms.

Proposition 3.1. There is a natural equivalence $D: 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$ of the identity functor, given by

$$D_X: X \rightarrow X, \quad D_X := \sum_{\ell=1}^{n^X} \iota_{\ell}^X \circ \pi_{\ell}^X (\dim V_{j_{\ell}^X})^{-1} \quad (3.1)$$

for all objects $X \in |\mathcal{C}|$. Here j_{ℓ}^X , ι_{ℓ}^X and π_{ℓ}^X are as in Definition 2.1(2d) or Definition A.20(3c).

Corollary 3.2. For any two objects $X, Y \in |\mathcal{C}|$, the map

$$\varphi_{X,Y}: \text{Hom}(Y, X) \otimes \text{Hom}(X, Y) \rightarrow k, \quad f \otimes g \mapsto \text{tr}_X(D_X \circ f \circ g). \quad (3.2)$$

is a non-degenerate symmetric and associative k -bilinear form, i.e. it is a non-degenerate k -bilinear form and satisfies

- (1) Symmetry: $\varphi_{X,Y}(f \otimes g) = \varphi_{Y,X}(g \otimes f)$ for all morphisms $f: Y \rightarrow X$ and $g: X \rightarrow Y$ of \mathcal{C} , and
- (2) Associativity: $\varphi_{X,Z}(f \otimes (g \circ h)) = \varphi_{X,Y}((f \circ g) \otimes h)$ for all $f: Z \rightarrow X$, $g: Y \rightarrow Z$ and $h: X \rightarrow Y$.

Proof. This follows from the cyclic property of the trace, from non-degeneracy of \mathcal{C} (Definition A.25) and from the fact that D_X is invertible for all $X \in |\mathcal{C}|$. \square

It is then possible to write down a pair of dual bases of $\text{Hom}(X, Y)$ and $\text{Hom}(Y, X)$ with respect to $\varphi_{X,Y}$.

Proposition 3.3. Let $X, Y \in |\mathcal{C}|$. Then

$$\{e_{\alpha\beta} = \iota_{\alpha}^Y \circ \pi_{\beta}^X: X \rightarrow Y \mid 1 \leq \alpha \leq n^Y, 1 \leq \beta \leq n^X, j_{\alpha}^Y = j_{\beta}^X\}, \quad (3.3)$$

$$\{e^{\gamma\delta} = \iota_{\delta}^X \circ \pi_{\gamma}^Y: X \rightarrow Y \mid 1 \leq \gamma \leq n^Y, 1 \leq \delta \leq n^X, j_{\gamma}^Y = j_{\delta}^X\} \quad (3.4)$$

form a pair of dual basis of $\text{Hom}(X, Y)$ and $\text{Hom}(Y, X)$ with respect to $\varphi_{X,Y}$, i.e.

$$\varphi_{X,Y}(e^{\gamma\delta} \otimes e_{\alpha\beta}) = \delta_{\alpha\gamma} \delta_{\beta\delta}. \quad (3.5)$$

Here we have used Definition 2.1(2d) (Definition A.20(3c)) for both X and Y .

Proof. Use Proposition 2.2(4) for both X and Y . \square

3.2 The long forgetful functor

Definition 3.4. The *universal object* of \mathcal{C} is defined as

$$\widehat{V} := \bigoplus_{j \in I} V_j \quad (3.6)$$

Note that the universal object is determined up to isomorphism by the category \mathcal{C} and that it is determined fully as soon as a family $\{V_j\}_{j \in I}$ and a total order of I have been fixed. We assume from now on that such a choice has been made.

Proposition 3.5. (1) The k -vector space $R = \text{End}(\widehat{V})$ forms a commutative separable k -algebra with respect to composition.

(2) A basis $(\lambda_j)_{j \in I}$ of orthogonal idempotents for R is given by $\lambda_j(v) = 0$ if $v \in V_\ell$, $\ell \neq j$, and $\lambda_j(v) = v$ if $v \in V_j$.

(3) Every morphism $f: \widehat{V} \rightarrow \widehat{V}$ is of the form $f = \sum_{j \in I} f_j \lambda_j$ with some $f_j \in k$.

Proof. Definition 2.1(2a) and Proposition 2.2(4). \square

The following definition is the canonical functor $\widehat{\omega}: \mathcal{C} \rightarrow {}_R\mathcal{M}_R$ of [24] composed with the forgetful functor ${}_R\mathcal{M}_R \rightarrow \mathbf{Vect}_k$.

Definition 3.6. The *long forgetful functor* is the functor

$$\begin{aligned} \omega: \mathcal{C} &\rightarrow \mathbf{Vect}_k, & X &\mapsto \text{Hom}(\widehat{V}, \widehat{V} \otimes X), \\ & & f &\mapsto (\text{id}_{\widehat{V}} \otimes f) \circ -. \end{aligned} \quad (3.7)$$

Note that the long forgetful functor is k -linear and takes values in \mathbf{fdVect}_k .

Proposition 3.7. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then $\omega(X)$, $X \in |\mathcal{C}|$, has a left-dual $(\omega(X)^*, \text{ev}_{\omega(X)}, \text{coev}_{\omega(X)})$ where $\omega(X)^* = \text{Hom}(\widehat{V} \otimes X, \widehat{V})$,

$$\text{ev}_{\omega(X)}: \omega(X)^* \otimes \omega(X) \rightarrow k, \quad \vartheta \otimes v \mapsto \text{tr}_{\widehat{V}}(D_{\widehat{V}} \circ \vartheta \circ v), \quad (3.8)$$

$$\text{coev}_{\omega(X)}: k \rightarrow \omega(X) \otimes \omega(X)^*, \quad 1 \mapsto \sum_j e_j^{(X)} \otimes e_{(X)}^j. \quad (3.9)$$

Here, $\{e_j^{(X)}\}_j$ and $\{e_{(X)}^j\}_j$ denote a pair of dual bases of $\omega(X) = \text{Hom}(\widehat{V}, \widehat{V} \otimes X)$ and $\omega(X)^* = \text{Hom}(\widehat{V} \otimes X, \widehat{V})$ with respect to $\varphi_{\widehat{V}, \widehat{V} \otimes X}$. Given any morphism $f: X \rightarrow Y$ of \mathcal{C} , the morphism dual to $\omega(f) = (\text{id}_{\widehat{V}} \otimes f) \circ -$ is given by

$$\omega(f)^* = - \circ (\text{id}_{\widehat{V}} \otimes f). \quad (3.10)$$

Proof. Corollary 3.2 and Proposition 3.3 imply the triangle identities. \square

Proposition 3.8. The long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ is faithful.

Proof. Let $X, Y \in |\mathcal{C}|$ and $f, g: X \rightarrow Y$ be arbitrary morphisms of \mathcal{C} . We have to show that $\omega(f) = \omega(g)$ implies $f = g$.

Choose some arbitrary $\ell \in I$, $q: V_\ell \rightarrow X$ and $p: Y \rightarrow V_\ell$. For every $\ell \in I$, denote by $\iota_\ell^{(\widehat{V})}: V_\ell \rightarrow \widehat{V}$ and $\pi_\ell^{(\widehat{V})}: \widehat{V} \rightarrow V_\ell$ the morphisms of Definition 2.1(2d) (Definition A.20(3c))

associated with V_ℓ for $X = \widehat{V}$. Then define $v := (\iota_0^{(\widehat{V})} \otimes q) \circ \lambda_{V_\ell}^{-1} \circ \pi_\ell^{(\widehat{V})} : \widehat{V} \rightarrow \widehat{V} \otimes X$ and $\eta := \iota_\ell^{(\widehat{V})} \circ \lambda_{V_\ell} \circ (\pi_0^{(\widehat{V})} \otimes p) : \widehat{V} \otimes Y \rightarrow \widehat{V}$. We compute that

$$0 = \eta \circ (\omega(f) - \omega(g))(v) = \eta \circ (\text{id}_{\widehat{V}} \otimes (f - g)) \circ v \quad (3.11)$$

and

$$0 = \text{tr}_{\widehat{V}}(\eta \circ (\text{id}_{\widehat{V}} \otimes (f - g)) \circ v) = \text{tr}_{V_\ell}(p \circ (f - g) \circ q). \quad (3.12)$$

This holds for any $\ell \in I$ and any p and q . If we insert all $q = \iota_m^{(X)}$ and $p = \pi_n^{(Y)}$ with $j_m = j_n = \ell$, we conclude that $0 = f - g$. \square

Remark 3.9. Our definition of a modular category does not assume the existence of all finite limits (preabelian category) nor that all monomorphisms and all epimorphisms are normal (abelian category). In Corollary 6.7 below, we nevertheless see that all finitely semisimple additive ribbon categories with $k = \text{End}(\mathbb{1})$ a field and therefore all modular categories are in fact abelian and that the long forgetful functor is exact.

The remainder of the present subsection can be skipped on first reading. The results are, however, needed in several proofs below.

Lemma 3.10. Let $X \in |\mathcal{C}|$. Then there are natural isomorphisms

$$\begin{aligned} \Phi_X : \omega(X) &\rightarrow \omega(X^*)^*, \\ v &\mapsto D_{\widehat{V}}^{-1} \circ \rho_{\widehat{V}} \circ (\text{id}_{\widehat{V}} \otimes \overline{\text{ev}}_X) \circ \alpha_{\widehat{V}, X, X^*} \circ (v \otimes \text{id}_{X^*}) \circ (D_{\widehat{V}} \otimes \text{id}_{X^*}), \end{aligned} \quad (3.13)$$

$$\begin{aligned} \Psi_X : \omega(X)^* &\rightarrow \omega(X^*), \\ \vartheta &\mapsto (\vartheta \otimes \text{id}_{X^*}) \circ \alpha_{\widehat{V}, X, X^*}^{-1} \circ (\text{id}_{\widehat{V}} \otimes \text{coev}_X) \circ \rho_{\widehat{V}}^{-1}. \end{aligned} \quad (3.14)$$

Their composites are given by

$$\begin{aligned} \Xi_X := \Psi_{X^*} \circ \Phi_X : \omega(X) &\rightarrow \omega(X^{**}), \\ v &\mapsto (D_{\widehat{V}}^{-1} \otimes \tau_X) \circ v \circ D_{\widehat{V}}, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \Theta_X := \Phi_{X^*} \circ \Psi_X : \omega(X)^* &\rightarrow \omega(X^{**})^*, \\ \vartheta &\mapsto D_{\widehat{V}}^{-1} \circ \vartheta \circ (D_{\widehat{V}} \otimes \tau_X^{-1}). \end{aligned} \quad (3.16)$$

Here $\tau_X : X \rightarrow X^{**}$ denotes the isomorphism of (A.24).

Proof. Naturality follows from the properties of dual morphisms. The morphisms Φ_X and Ψ_X are invertible because $D_{\widehat{V}}$ is and because of the triangle identities (A.12) to (A.15). In order to determine their composites, one needs (A.24), (A.25) and (A.26). \square

Proposition 3.11. Let $X \in |\mathcal{C}|$, and let $\{e_j^{(X)}\}_j$ and $\{e_{(X)}^j\}_j$ form a pair of dual bases of $\omega(X)$ and $\omega(X)^*$ with respect to $\varphi_{\widehat{V}, \widehat{V} \otimes X}$. Then

$$\sum_j e_j^{(X)} \circ e_{(X)}^j = \text{id}_{\widehat{V} \otimes X} \quad (3.17)$$

and

$$\sum_j \Psi(e_{(X)}^j) \circ \Phi(e_j^{(X)}) = \text{id}_{\widehat{V} \otimes X^*}. \quad (3.18)$$

Proof. Show (3.17) first for the pair of dual bases of Proposition 3.3 with \widehat{V} instead of X and $\widehat{V} \otimes X$ instead of Y . Then the claim follows for any other pair of dual bases. In order to verify (3.18), show that $(\Psi(e_{(X)}^j))_j$ and $(\Phi(e_j^{(X)}))_j$ form a pair of dual bases of $\omega(X^*)$ and $\omega(X^*)^*$, respectively. Then the claim follows from (3.17). \square

Proposition 3.12. Let $X, Y \in |\mathcal{C}|$, and let $\{e_j^{(X)}\}_j$ and $\{e_{(X)}^j\}_j$ as well as $\{e_\ell^{(Y)}\}_\ell$ and $\{e_{(Y)}^\ell\}_\ell$ be pairs of dual bases of $\omega(X)$ and $\omega(X)^*$ as well as of $\omega(Y)$ and $\omega(Y)^*$, respectively. Define

$$e_{j\ell}^{(X \otimes Y)} := \alpha_{\widehat{V}, X, Y} \circ (e_j^{(X)} \otimes \text{id}_Y) \circ e_\ell^{(Y)} \in \omega(X \otimes Y), \quad (3.19)$$

$$e_{(X \otimes Y)}^{j\ell} := e_{(Y)}^\ell \circ (e_{(X)}^j \otimes \text{id}_Y) \circ \alpha_{\widehat{V}, X, Y}^{-1} \in \omega(X \otimes Y)^*. \quad (3.20)$$

Then $(\omega(X \otimes Y)^*, \text{ev}_{\omega(X \otimes Y)}, \text{coev}_{\omega(X \otimes Y)})$ is a left-dual of $\omega(X \otimes Y)$ with

$$\text{ev}_{\omega(X \otimes Y)}: \omega(X \otimes Y)^* \otimes \omega(X \otimes Y) \rightarrow k, \quad \vartheta \otimes v \mapsto \varphi_{\widehat{V}, \widehat{V} \otimes (X \otimes Y)}(\vartheta \otimes v), \quad (3.21)$$

$$\text{coev}_{\omega(X \otimes Y)}: k \rightarrow \omega(X \otimes Y) \otimes \omega(X \otimes Y)^*, \quad 1 \mapsto \sum_{j, \ell} e_{j\ell}^{(X \otimes Y)} \otimes e_{(X \otimes Y)}^{j\ell}. \quad (3.22)$$

Proof. The triangle identities for $\text{ev}_{\omega(X \otimes Y)}$ and $\text{coev}_{\omega(X \otimes Y)}$ follow from the triangle identities for $\text{ev}_{\omega(X)}$ and $\text{coev}_{\omega(X)}$ as well as for $\text{ev}_{\omega(Y)}$ and $\text{coev}_{\omega(Y)}$, c.f. Proposition 3.7, and from (3.17). Note that the $e_{j\ell}^{(X \otimes Y)}$ are in general linearly dependent. The sum in (3.22) nevertheless yields a perfectly acceptable coevaluation map. \square

3.3 Functors with separable Frobenius structure

We can now show that the long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ satisfies the following conditions due to Szlachányi [25].

Definition 3.13. Let \mathcal{D} and \mathcal{D}' be monoidal categories. A functor with separable Frobenius structure $(F, F_{X,Y}, F_0, F^{X,Y}, F^0): \mathcal{D} \rightarrow \mathcal{D}'$ is a functor $F: \mathcal{D} \rightarrow \mathcal{D}'$ which is lax monoidal as $(F, F_{X,Y}, F_0)$ and oplax monoidal as $(F, F^{X,Y}, F^0)$ (c.f. Definition A.2) and which satisfies the following compatibility conditions.

$$F_{X,Y} \circ F^{X,Y} = \text{id}_{F(X \otimes Y)}, \quad (3.23)$$

$$\begin{array}{ccccc} F(X \otimes Y) \otimes' FZ & \xrightarrow{F_{X \otimes Y, Z}} & F((X \otimes Y) \otimes Z) & \xrightarrow{F\alpha_{X,Y,Z}} & F(X \otimes (Y \otimes Z)) \\ \downarrow F^{X,Y} \otimes' \text{id}_{FZ} & & & & \downarrow F^{X,Y \otimes Z} \\ (FX \otimes' FY) \otimes' FZ & \xrightarrow{\alpha'_{FX, FY, FZ}} & FX \otimes' (FY \otimes' FZ) & \xrightarrow{\text{id}_{FX} \otimes' F_{Y,Z}} & FX \otimes' F(Y \otimes Z), \end{array} \quad (3.24)$$

$$\begin{array}{ccccc}
FX \otimes' F(Y \otimes Z) & \xrightarrow{F_{X,Y \otimes Z}} & F(X \otimes (Y \otimes Z)) & \xrightarrow{F\alpha_{X,Y,Z}^{-1}} & F((X \otimes Y) \otimes Z) \\
\downarrow \text{id}_{FX} \otimes' F^{Y,Z} & & & & \downarrow F^{X \otimes Y, Z} \\
FX \otimes' (FY \otimes' FZ) & \xrightarrow{\alpha_{FX,FY,FZ}^{-1}} & (FX \otimes' FY) \otimes' FZ & \xrightarrow{F_{X,Y} \otimes' \text{id}_{FZ}} & F(X \otimes Y) \otimes' FZ,
\end{array} \tag{3.25}$$

for all $X, Y, Z \in |\mathcal{D}|$.

The reason for choosing the term *Frobenius structure* becomes obvious if one visualizes the compatibility conditions by the following diagrams. For more details on these diagrams, we refer to [31–33].

$$\begin{array}{ccc}
\begin{array}{c} X \otimes Y \\ \text{cup} \\ X \otimes Y \end{array} = \begin{array}{c} X \otimes Y \\ \text{id} \\ X \otimes Y \end{array} &
\begin{array}{c} X \quad Y \otimes Z \\ \text{cup} \\ X \otimes Y \quad Z \end{array} = \begin{array}{c} X \quad Y \otimes Z \\ \text{id} \\ X \otimes Y \quad Z \end{array} &
\begin{array}{c} X \otimes Y \quad Z \\ \text{cup} \\ X \quad Y \otimes Z \end{array} = \begin{array}{c} X \otimes Y \quad Z \\ \text{id} \\ X \quad Y \otimes Z \end{array}
\end{array} \tag{3.26}$$

Theorem 3.14. The long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ has a separable Frobenius structure $(\omega, \omega_{X,Y}, \omega_0, \omega^{X,Y}, \omega^0)$ with

$$\omega_{X,Y}: \omega(X) \otimes \omega(Y) \rightarrow \omega(X \otimes Y), \quad f \otimes g \mapsto \alpha_{\widehat{V}, X, Y} \circ (f \otimes \text{id}_Y) \circ g, \tag{3.27}$$

$$\omega_0: k \rightarrow \omega(\mathbb{1}), \quad 1 \mapsto \rho_{\widehat{V}}^{-1}, \tag{3.28}$$

and

$$\omega^{X,Y}: \omega(X \otimes Y) \rightarrow \omega(X) \otimes \omega(Y), \quad h \mapsto \sum_{j,\ell} \text{ev}_{\omega(X \otimes Y)}(e_{(X \otimes Y)}^{j\ell} \otimes h) e_j^{(X)} \otimes e_\ell^{(Y)}, \tag{3.29}$$

$$\omega^0: \omega(\mathbb{1}) \rightarrow k, \quad v \mapsto \text{ev}_{\omega(\mathbb{1})}(\rho_{\widehat{V}} \otimes v), \tag{3.30}$$

using the $e_{(X \otimes Y)}^{j\ell}$ of Proposition 3.12.

Proof. We need to verify the following.

- (1) ω is indeed a functor.
- (2) $\omega_{X,Y}$ and $\omega^{X,Y}$ are natural transformations.
- (3) $(\omega, \omega_{X,Y}, \omega_0)$ is lax monoidal. The hexagon axiom for the lax monoidal functor follows from the pentagon axiom in \mathcal{C} , and the two squares follow from the triangle axiom.
- (4) $(\omega, \omega^{X,Y}, \omega^0)$ is oplax monoidal. Again, the hexagon axiom follows from the pentagon, and the two squares from the triangle.
- (5) In order to show the compatibility conditions (3.24) and (3.25), we verify for each $h \in \omega(X \otimes Y)$ and $w \in \omega(Z)$ that

$$\begin{aligned}
& \omega^{X,Y \otimes Z} \circ \omega(\alpha_{X,Y,Z}) \circ \omega_{X \otimes Y, Z}(h \otimes w) \\
&= (\text{id}_{\omega(X)} \otimes \omega_{Y,Z}) \circ \alpha_{\omega(X), \omega(Y), \omega(Z)} \circ (\omega^{X,Y} \otimes \text{id}_{\omega(Z)})(h \otimes w),
\end{aligned} \tag{3.31}$$

which follows from the definitions, using the left-duals of Proposition 3.7, the basis of Proposition 3.12 and the pentagon axiom of \mathcal{C} . Similarly, for $v \in \omega(X)$ and $f \in \omega(Y \otimes Z)$, we verify that

$$\begin{aligned} & \omega^{X \otimes Y, Z} \circ \omega(\alpha_{X, Y, Z}^{-1}) \circ \omega_{X, Y \otimes Z}(v \otimes f) \\ &= (\omega_{X, Y} \otimes \text{id}_{\omega(Z)}) \circ \alpha_{\omega(X), \omega(Y), \omega(Z)}^{-1} \circ (\text{id}_{\omega(X)} \otimes \omega^{Y, Z})(v \otimes f). \end{aligned} \quad (3.32)$$

The condition (3.23) follows from Proposition 3.12. □

Remark 3.15. Under the conditions of Theorem 3.14,

$$\omega^0 \circ \omega_0 = |I|, \quad (3.33)$$

where $|I|$ is the number of (isomorphism classes of) simple objects as in Definition 2.1(2) or Definition A.20(3).

If the characteristic of k divides $|I|$, then this is zero. Otherwise, ω_0 is a split monomorphism with left-inverse $\omega^0/|I|$. Recall that because of (3.23), the $\omega_{X, Y}$ are split epimorphisms with right-inverse $\omega^{X, Y}$. If the characteristic of k does not divide $|I|$, one may call the Frobenius structure *special*, c.f. [34, Definition 2.3]. It is interesting to note that the question of whether the right hand side of (3.33) is zero or not, does not play any role in the following.

4 Tannaka–Kreĭn reconstruction

In this section, unless specified otherwise, \mathcal{C} denotes a finitely semisimple additive ribbon category for which $k = \text{End}(\mathbb{1})$ is a field.

4.1 Coalgebra structure

If $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ is the long forgetful functor, then the coend $H = \mathbf{coend}(\mathcal{C}, \omega)$ has the structure of a coassociative counital coalgebra (H, Δ, ε) as in Section 2.4.

For the homogeneous elements of the coend (2.22), we write $[\vartheta|v]_X \in \omega(X)^* \otimes \omega(X)$ with $\vartheta \in \omega(X)^*$ and $v \in \omega(X)$, i.e. $\vartheta: \widehat{V} \otimes X \rightarrow \widehat{V}$ and $v: \widehat{V} \rightarrow \widehat{V} \otimes X$, using the left-duals of Proposition 3.7. The relations in the quotient (2.22) then read,

$$[\zeta \circ (\text{id}_{\widehat{V}} \otimes f)|v]_X = [\zeta|(\text{id}_{\widehat{V}} \otimes f) \circ v]_Y, \quad (4.1)$$

where $v: \widehat{V} \rightarrow \widehat{V} \otimes X$, $f: X \rightarrow Y$ and $\zeta: \widehat{V} \otimes Y \rightarrow \widehat{V}$. The coalgebra operations of H can be written as

$$\Delta([\vartheta|v]_X) = \sum_j [\vartheta|e_j^{(X)}]_X \otimes [e_{(X)}^j|v]_X, \quad (4.2)$$

$$\varepsilon([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes v), \quad (4.3)$$

and the universal coaction as

$$\delta_X^\omega(v) = \sum_j e_j^{(X)} \otimes [e_{(X)}^j|v]_X \quad (4.4)$$

for all $v: \widehat{V} \rightarrow \widehat{V} \otimes X$ and $\vartheta: \widehat{V} \otimes X \rightarrow \widehat{V}$, $X \in |\mathcal{C}|$.

4.2 Semisimplicity

With the explicit description of the coend of Section 2.4, we can show that the coend $H = \mathbf{coend}(\mathcal{C}, \omega)$ is a finite-dimensional split cosemisimple coalgebra.

Proposition 4.1. Let \mathcal{D} be an **Ab**-enriched ribbon category, $k = \text{End}(\mathbb{1})$ and $\omega: \mathcal{D} \rightarrow \mathbf{Vect}_k$ be a k -linear functor taking values in \mathbf{fdVect}_k .

- (1) If \mathcal{D} is semisimple with a family $\{V_j\}_j$, $j \in I$, of simple objects as in Definition A.20(3), then

$$\mathbf{coend}(\mathcal{D}, \omega) \cong \coprod_{j \in I} \omega(V_j)^* \otimes \omega(V_j). \quad (4.5)$$

With the operations (4.2) and (4.3), the coend therefore forms a split cosemisimple coalgebra.

- (2) If \mathcal{D} is finitely semisimple, then the coend is finite-dimensional, i.e.

$$\mathbf{coend}(\mathcal{D}, \omega) \cong \bigoplus_{j \in I} \omega(V_j)^* \otimes \omega(V_j), \quad (4.6)$$

where the coproduct has turned into a finite biproduct.

Proof. We show that the composition of the inclusion

$$\bigoplus_{j \in I} \omega(V_j)^* \otimes \omega(V_j) \rightarrow \coprod_{X \in |\mathcal{D}|} \omega(X)^* \otimes \omega(X) \quad (4.7)$$

with the canonical projection

$$\coprod_{X \in |\mathcal{D}|} \omega(X)^* \otimes \omega(X) \rightarrow \left(\coprod_{X \in |\mathcal{D}|} \omega(X)^* \otimes \omega(X) \right) / N \quad (4.8)$$

is a bijection. In order to see that it is surjective, we use Definition 2.1(2d) or Definition A.20(3c), and in order to see that it is injective, Proposition 2.2(5). \square

4.3 Algebra structure

In this section, we use the monoidal structure of \mathcal{C} in order to equip the coend with the structure of an associative unital algebra. In the remainder of Section 4, all commutative diagrams are in \mathbf{Vect}_k . We write k for the monoidal unit object of \mathbf{Vect}_k .

In order to reconstruct an algebra structure on $H = \mathbf{coend}(\mathcal{C}, \omega)$ from the monoidal structure of \mathcal{C} , we consider the category \mathcal{C} with the functor $\omega \otimes \omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$, $X \mapsto \omega(X) \otimes \omega(X)$, $f \mapsto \omega(f) \otimes \omega(f)$. The corresponding coend and the universal coaction are given as follows.

Proposition 4.2 (see, for example [13]). Let \mathcal{D} be a monoidal category, $\omega: \mathcal{D} \rightarrow \mathbf{Vect}_k$ be a functor taking values in \mathbf{fdVect}_k and $H = \mathbf{coend}(\mathcal{D}, \omega)$.

- (1) The coend of $\omega \otimes \omega: \mathcal{D} \times \mathcal{D} \rightarrow \mathbf{Vect}_k$ is the tensor product coalgebra,

$$H \otimes H \cong \mathbf{coend}(\mathcal{D} \times \mathcal{D}, \omega \otimes \omega), \quad (4.9)$$

with the operations $\Delta_{H \otimes H} = (\text{id}_H \otimes \sigma_{H, H} \otimes \text{id}_H) \circ (\Delta_H \otimes \Delta_H)$ and $\varepsilon_{H \otimes H} = \varepsilon_H \otimes \varepsilon_H$.

- (2) The corresponding universal coaction is given by $\delta_{X,Y}^{\omega \otimes \omega} : \omega(X) \otimes \omega(Y) \rightarrow (\omega(X) \otimes \omega(Y)) \otimes (H \otimes H)$ where

$$\delta_{X,Y}^{\omega \otimes \omega} = (\text{id}_{\omega(X)} \otimes \sigma_{H, \omega(Y)} \otimes \text{id}_H) \circ (\delta_X^\omega \otimes \delta_Y^\omega). \quad (4.10)$$

In addition, let $\mathbf{1}$ denote the monoidal category whose only morphism is the identity of the monoidal unit. Then $\mathbf{1}$ with the functor $\omega^{\otimes 0} : \mathbf{1} \rightarrow \mathbf{Vect}_k$, $*$ $\mapsto k$, $\text{id}_* \mapsto \text{id}_k$, has the trivial coalgebra as the coend, $\mathbf{coend}(\mathbf{1}, \omega^{\otimes 0}) \cong k$, and the universal coaction $\delta^{\omega^{\otimes 0}} : k \rightarrow k \otimes k$, $\delta^{\omega^{\otimes 0}} = \rho_k^{-1}$.

Theorem 4.3. Let $\omega : \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then the coend $H = \mathbf{coend}(\mathcal{C}, \omega)$ is equipped with the structure (H, μ, η) of an associative unital algebra. Its operations are determined from the universal property of the coend by commutativity of

$$\begin{array}{ccc} \omega(X) \otimes \omega(Y) & \xrightarrow{\delta_{X,Y}^{\omega \otimes \omega}} & (\omega(X) \otimes \omega(Y)) \otimes (H \otimes H) \\ \omega_{X,Y} \downarrow & \searrow & \downarrow \text{id}_{\omega(X) \otimes \omega(Y)} \otimes \mu \\ \omega(X \otimes Y) & & \\ \delta_{X \otimes Y}^\omega \downarrow & \searrow & \downarrow \\ \omega(X \otimes Y) \otimes H & \xrightarrow{\omega_{X,Y} \otimes \text{id}_H} & (\omega(X) \otimes \omega(Y)) \otimes H \end{array} \quad (4.11)$$

and of

$$\begin{array}{ccc} k & \xrightarrow{\delta^{\omega^{\otimes 0}}} & k \otimes k \\ \omega_0 \downarrow & \searrow & \downarrow \text{id}_k \otimes \eta \\ \omega(\mathbb{1}) & & \\ \delta_{\mathbb{1}}^\omega \downarrow & \searrow & \downarrow \\ \omega(\mathbb{1}) \otimes H & \xrightarrow{\omega^0 \otimes \text{id}_H} & k \otimes H. \end{array} \quad (4.12)$$

In order to prove the theorem, it is convenient to compute the operations μ and η in a basis of $H = \mathbf{coend}(\mathcal{C}, \omega)$ that is adapted to the matrix coalgebra structure (4.6) as follows.

Lemma 4.4. Under the conditions of Theorem 4.3, the operations are given by

$$\mu([\vartheta|v]_X \otimes [\zeta|w]_Y) = [\zeta \circ (\vartheta \otimes \text{id}_Y) \circ \alpha_{\widehat{V}, X, Y}^{-1} | \alpha_{\widehat{V}, X, Y} \circ (v \otimes \text{id}_Y) \circ w]_{X \otimes Y}, \quad (4.13)$$

$$\eta(1) = [\rho_{\widehat{V}} | \rho_{\widehat{V}}^{-1}]_{\mathbb{1}} \quad (4.14)$$

for $\vartheta \in \omega(X)^*$, $\zeta \in \omega(Y)^*$, $v \in \omega(X)$ and $w \in \omega(Y)$.

Proof. In order to show that the operations (4.13) and (4.14) make the diagrams (4.11) and (4.12) commute, one needs the definitions and Proposition 3.12. \square

Proof of Theorem 4.3. We use the operations μ and η as given in Lemma 4.4. In order to show that μ is well defined on the quotient modulo N of (2.22), one needs the relations (4.1). For associativity of μ , one needs the pentagon axiom for the associator of \mathcal{C} , and for the unit laws the triangle axiom. \square

In this section, we have not only used that \mathbf{Vect}_k is small cocomplete and that \otimes preserves colimits in both arguments, but also that \mathbf{Vect}_k has a symmetric braiding.

4.4 Weak Hopf Algebra structure

In this section, we show that the algebra and coalgebra structure of the coend satisfy the compatibility conditions of a WBA, and we use the left-duals in \mathcal{C} in order to construct an antipode that turns it into a WHA.

Theorem 4.5. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then the coend $H = \mathbf{coend}(\mathcal{C}, \omega)$ has the structure of a WBA $(H, \mu, \eta, \Delta, \varepsilon)$.

Proof. In order to verify the conditions of Definition 2.3, we express the operations of H in the form (4.13), (4.14), (4.2) and (4.3). In order to verify (2.4), we use Proposition 3.12. For (2.5) and (2.6), we need the triangle equations for the evaluation and coevaluation maps of Proposition 3.7 as well as the associativity property of Corollary 3.2 for the traces involved. \square

In order to reconstruct an antipode from duality in \mathcal{C} , we consider the opposite category \mathcal{C}^{op} with the functor $\omega^*: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Vect}_k$, $X \mapsto \omega(X)^*$, $f \mapsto \omega(f)^*$. The corresponding coend and the universal coaction on the duals are characterized by the following result.

Proposition 4.6 (see, for example [13]). Let \mathcal{D} be a left-autonomous monoidal category, $\omega: \mathcal{D} \rightarrow \mathbf{Vect}_k$ be a functor taking values in \mathbf{fdVect}_k and $H = \mathbf{coend}(\mathcal{D}, \omega)$.

- (1) The coend of $\omega^*: \mathcal{D}^{\text{op}} \rightarrow \mathbf{Vect}_k$ is the coopposite coalgebra,

$$H^{\text{cop}} \cong \mathbf{coend}(\mathcal{D}^{\text{op}}, \omega^*). \quad (4.15)$$

- (2) The corresponding universal coaction is given by $\delta_X^{\omega^*}: \omega(X)^* \rightarrow \omega(X)^* \otimes H^{\text{cop}}$ where

$$\begin{aligned} \delta_X^{\omega^*} &= \sigma_{H, \omega(X)^*} \circ (\lambda_H \otimes \text{id}_{\omega(X)^*}) \circ ((\text{ev}_{\omega(X)} \otimes \text{id}_H) \otimes \text{id}_{\omega(X)^*}) \\ &\quad \circ (\alpha_{\omega(X)^*, \omega(X), H}^{-1} \otimes \text{id}_{\omega(X)^*}) \circ ((\text{id}_{\omega(X)^*} \otimes \delta_X^{\omega}) \otimes \text{id}_{\omega(X)^*}) \\ &\quad \circ \alpha_{\omega(X)^*, \omega(X), \omega(X)^*}^{-1} \circ (\text{id}_{\omega(X)^*} \otimes \text{coev}_{\omega(X)}) \circ \rho_{\omega(X)^*}^{-1}. \end{aligned} \quad (4.16)$$

Theorem 4.7. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then the coend $H = \mathbf{coend}(\mathcal{C}, \omega)$ has the structure of a WHA $(H, \mu, \eta, \Delta, \varepsilon, S)$. Here the antipode $S: H \rightarrow H$ is determined from the universal property of the coend by commutativity of

$$\begin{array}{ccc} \omega(X)^* & \xrightarrow{\delta_X^{\omega^*}} & \omega(X)^* \otimes H \\ \Psi_X \downarrow & \searrow & \downarrow \text{id}_{\omega(X)^*} \otimes S \\ \omega(X^*) & & \\ \delta_{X^*}^{\omega} \downarrow & & \\ \omega(X^*) \otimes H & \xrightarrow{\Psi_X^{-1} \otimes \text{id}_H} & \omega(X)^* \otimes H. \end{array} \quad (4.17)$$

with Ψ_X as in Lemma 3.10.

Again, we first express the antipode in our preferred basis.

Lemma 4.8. Under the conditions of Theorem 4.7, the antipode is given by

$$S([\vartheta|v]_X) = [\Phi_X(v)|\Psi_X(\vartheta)]_{X^*} \quad (4.18)$$

with Φ and Ψ as defined in Lemma 3.10.

Proof. We verify in a direct computation that the linear map S of (4.18) makes the diagram (4.17) commute. The top right of that diagram can be computed from the definitions using the left-duals of Proposition 3.7 whereas the bottom left can be obtained from the explicit expression for Ψ in Lemma 3.10. \square

Proof of Theorem 4.7. Using the expression (4.18), we can employ the relations (4.1) to show that S is well defined on the quotient (2.22). Before we verify (2.11) and (2.12), we first compute ε_t and ε_s ,

$$\varepsilon_t([\vartheta|v]_X) = [\Phi_X(v) \circ \Psi_X(\vartheta) \circ \rho_{\widehat{V}}| \rho_{\widehat{V}}^{-1}]_1, \quad (4.19)$$

$$\varepsilon_s([\vartheta|v]_X) = [\rho_{\widehat{V}}| \rho_{\widehat{V}}^{-1} \circ \vartheta \circ v]_1, \quad (4.20)$$

for all $\vartheta \in \omega(X)^*$ and $v \in \omega(X)$. In order to verify (2.11), one needs (3.17), and for (2.12), one needs the triangle and pentagon axioms in \mathcal{C} as well as (3.18). Finally, (2.13) can be obtained from (2.12) and (3.17). \square

Proposition 4.9. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor and $H = \mathbf{coend}(\mathcal{C}, \omega)$ the reconstructed WHA. Then

$$S^2([\vartheta|v]_X) = [D_{\widehat{V}}^{-1} \circ \vartheta \circ (D_{\widehat{V}} \otimes \text{id}_X) | (D_{\widehat{V}}^{-1} \otimes \text{id}_X) \circ v \circ D_{\widehat{V}}]_X, \quad (4.21)$$

for all $v \in \omega(X)$ and $\vartheta \in \omega(X)^*$, $X \in |\mathcal{C}|$.

Proof. Using Lemma 4.8 and Lemma 3.10, we get

$$S^2([\vartheta|v]_X) = [\Theta_X(\vartheta) | \Xi_X(v)]_{X^{**}} \quad (4.22)$$

which implies the claim upon using the relations (4.1). \square

Remark 4.10. In the reconstructed WHA, we have

$$\varepsilon \circ \eta = \omega^0 \circ \omega_0 = |I| \in k, \quad (4.23)$$

c.f. Remark 3.15.

4.5 Coribbon structure

In this section, we define the notion of a coribbon WHA and show that the WHA reconstructed from \mathcal{C} has this structure.

Definition 4.11. Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a WHA. A linear form $f: H \rightarrow k$ is called

- (1) *convolution invertible* if there exists some linear $\overline{f}: H \rightarrow k$ such that $f(x')\overline{f}(x'') = \varepsilon(x) = \overline{f}(x')f(x'')$ for all $x \in H$,
- (2) *dual central* if $f(x')x'' = x'f(x'')$ for all $x \in H$,

(3) *dual group-like* if it is convolution invertible and

$$f(xy) = \varepsilon(x'y')f(x'')f(y'') = f(x')f(y')\varepsilon(x''y'') \quad (4.24)$$

for all $x, y \in H$.

Note that \bar{f} in (1) is uniquely determined by f . Every dual group-like linear form also satisfies $f(\varepsilon_t(x)) = \varepsilon(x) = f(\varepsilon_s(x))$ and $f(S(x)) = \bar{f}(x)$ for all $x \in H$.

Definition 4.12. A *coquasitriangular WHA* $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ over a field k is a WHA $(H, \mu, \eta, \Delta, \varepsilon, S)$ over k with a linear form $r: H \otimes H \rightarrow k$ (*universal r -form*) that satisfies the following conditions:

(1) For all $x, y \in H$,

$$r(x \otimes y) = \varepsilon(x'y')r(x'' \otimes y'') = r(x' \otimes y')\varepsilon(y''x'') \quad (4.25)$$

(2) There exists some linear $\bar{r}: H \otimes H \rightarrow k$ such that

$$\bar{r}(x' \otimes y')r(x'' \otimes y'') = \varepsilon(yx) \quad (4.26)$$

$$r(x' \otimes y')\bar{r}(x'' \otimes y'') = \varepsilon(xy) \quad (4.27)$$

(3) For all $x, y, z \in H$,

$$x'y'r(x'' \otimes y'') = r(x' \otimes y')y''x'' \quad (4.28)$$

$$r((xy) \otimes z) = r(y \otimes z')r(x \otimes z'') \quad (4.29)$$

$$r(x \otimes (yz)) = r(x' \otimes y)r(x'' \otimes z) \quad (4.30)$$

The WHA H is called *cotriangular* if in addition

$$r(x' \otimes y')r(y'' \otimes x'') = \varepsilon(xy) \quad (4.31)$$

for all $x, y \in H$.

Note that \bar{r} in (2) is uniquely determined by r if one imposes (4.25), (4.26) and (4.27). Condition (4.25) says that r is well defined on the tensor product $H \widehat{\otimes} H$ and on its opposite if H is viewed as the right-regular H -comodule. The conditions (4.26) and (4.27) express *weak convolution invertibility*, (4.28) *almost commutativity* and (4.29) and (4.30) compatibility with the tensor product.

Theorem 4.13. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then $H = \mathbf{coend}(\mathcal{C}, \omega)$ is a coquasitriangular WHA $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ whose universal r -form $r: H \otimes H \rightarrow k$ is determined from the universal property of the coend by commutativity of

$$\begin{array}{ccc}
 \omega(X) \otimes \omega(Y) & \xrightarrow{\delta_{X,Y}^{\omega \otimes \omega}} & (\omega(X) \otimes \omega(Y)) \otimes (H \otimes H) \\
 \sigma_{\omega(X), \omega(Y)} \downarrow & \searrow & \downarrow \text{id}_{\omega(X)} \otimes \text{id}_{\omega(Y)} \otimes r \\
 \omega(Y) \otimes \omega(X) & & \\
 \omega_{X,Y} \downarrow & & \\
 \omega(Y \otimes X) & & \\
 \omega(\sigma_{Y,X}) \downarrow & & \\
 \omega(X \otimes Y) & \xrightarrow{\omega_{X,Y}} \omega(X) \otimes \omega(Y) \xrightarrow{\rho_{\omega(X) \otimes \omega(Y)}^{-1}} & (\omega(X) \otimes \omega(Y)) \otimes k
 \end{array} \quad (4.32)$$

for all $X, Y \in |\mathcal{C}|$. Here, $\sigma_{Y,X}$ denotes the braiding of \mathcal{C} , $\sigma_{\omega(X),\omega(Y)}$ the braiding of \mathbf{Vect}_k and $\rho_{\omega(X) \otimes \omega(Y)}$ the right unit constraint of \mathbf{Vect}_k . Its weak convolution inverse $\bar{r}: H \otimes H \rightarrow k$ is determined by commutativity of

$$\begin{array}{ccc}
 \omega(X) \otimes \omega(Y) & \xrightarrow{\delta_{X,Y}^{\omega \otimes \omega}} & (\omega(X) \otimes \omega(Y)) \otimes (H \otimes H) \\
 \omega_{X,Y} \downarrow & \searrow & \downarrow \text{id}_{\omega(X)} \otimes \text{id}_{\omega(Y)} \otimes \bar{r} \\
 \omega(X \otimes Y) & & \\
 \omega(\sigma_{Y,X}^{-1}) \downarrow & & \\
 \omega(Y \otimes X) & & \\
 \omega^{Y,X} \downarrow & & \\
 \omega(Y) \otimes \omega(X) & \xrightarrow{\sigma_{\omega(Y),\omega(X)}} \omega(X) \otimes \omega(Y) \xrightarrow{\rho_{\omega(X) \otimes \omega(Y)}^{-1}} & (\omega(X) \otimes \omega(Y)) \otimes k
 \end{array} \tag{4.33}$$

for all $X, Y \in |\mathcal{C}|$.

Lemma 4.14. Under the conditions of Theorem 4.13, the universal r -form and its weak convolution inverse are given by

$$r([\vartheta|v]_X \otimes [\zeta|w]_Y) \tag{4.34}$$

$$= \text{ev}_{\omega(X \otimes Y)}((\zeta \circ (\vartheta \otimes \text{id}_Y) \circ \alpha_{\widehat{V},X,Y}^{-1}) \otimes ((\text{id}_{\widehat{V}} \otimes \sigma_{Y,X}) \circ \alpha_{\widehat{V},Y,X} \circ (w \otimes \text{id}_X) \circ v)),$$

$$\bar{r}([\vartheta|v]_X \otimes [\zeta|w]_Y) \tag{4.35}$$

$$= \text{ev}_{\omega(X \otimes Y)}((\vartheta \circ (\zeta \otimes \text{id}_X) \circ \alpha_{\widehat{V},Y,X}^{-1} \circ (\text{id}_{\widehat{V}} \otimes \sigma_{Y,X}^{-1})) \otimes (\alpha_{\widehat{V},X,Y} \circ (v \otimes \text{id}_Y) \circ w)).$$

Proof. We have to show that (4.34) and (4.35) make the diagrams (4.32) and (4.33) commute. In order to prove this, one needs the definitions of the morphisms that appear in these diagrams as well as Proposition 3.12. \square

Proof of Theorem 4.13. We verify in a direct computation that r and \bar{r} of (4.34) and (4.35) satisfy the conditions of Definition 4.12. In order to show (4.25), (4.26), (4.27) and (4.28), one needs Proposition 3.12 and the relations (4.1). In order to show (4.29) and (4.30), one needs in addition the pentagon axiom of the associator of \mathcal{C} , the hexagon axioms for the braiding of \mathcal{C} , and the cyclic property of the trace involved in $\text{ev}_{\omega(-)}$. \square

Definition 4.15. Let $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ be a coquasitriangular WHA. Then we define

(1) the linear form

$$q: H \otimes H \rightarrow k, \quad x \otimes y \mapsto r(x' \otimes y')r(y'' \otimes x''), \tag{4.36}$$

(2) the linear forms (*dual Drinfel'd elements*)

$$u: H \rightarrow k, \quad x \mapsto r(S(x'') \otimes x'), \tag{4.37}$$

$$v: H \rightarrow k, \quad x \mapsto r(S(x') \otimes x''). \tag{4.38}$$

Proposition 4.16. Under the assumptions of Theorem 4.13, the linear form $q: H \otimes H \rightarrow k$ is the unique linear map making the diagram

$$\begin{array}{ccc}
 \omega(X) \otimes \omega(Y) & \xrightarrow{\delta_{X,Y}^{\omega \otimes \omega}} & (\omega(X) \otimes \omega(Y)) \otimes (H \otimes H) \\
 \omega_{X,Y} \downarrow & \searrow & \downarrow \text{id}_{\omega(X)} \otimes \text{id}_{\omega(Y)} \otimes q \\
 \omega(X \otimes Y) & & \\
 Q_{X,Y} \downarrow & & \\
 \omega(X \otimes Y) & \xrightarrow{\omega_{X,Y}} \omega(X) \otimes \omega(Y) \xrightarrow{\rho_{\omega(X) \otimes \omega(Y)}^{-1}} & (\omega(X) \otimes \omega(Y)) \otimes k
 \end{array} \quad (4.39)$$

commute. Here $Q_{X,Y} = \sigma_{Y,X} \circ \sigma_{X,Y}: X \otimes Y \rightarrow X \otimes Y$, $X, Y \in |\mathcal{C}|$.

Proof. From (4.34), one can calculate using Proposition 3.12 that

$$\begin{aligned}
 & q([\vartheta|v]_X \otimes [\zeta|w]_Y) \\
 = & \text{ev}_{\omega(X \otimes Y)}((\zeta \circ (\vartheta \otimes \text{id}_Y) \circ \alpha_{\widehat{V},X,Y}^{-1}) \otimes ((\text{id}_{\widehat{V}} \otimes Q_{X,Y}) \circ \alpha_{\widehat{V},X,Y} \circ (v \otimes \text{id}_Y) \circ w)).
 \end{aligned} \quad (4.40)$$

Then commutativity of (4.39) can be verified in a direct computation. \square

Definition 4.17. A *coribbon WHA* $(H, \mu, \eta, \Delta, \varepsilon, S, r, \nu)$ over a field k is a coquasitriangular WHA $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ over k with a convolution invertible and dual central linear form $\nu: H \rightarrow k$ (*universal ribbon twist*) that satisfies the following conditions:

$$\nu(xy) = \nu(x')\nu(y')r(x'' \otimes y'')r(y''' \otimes x'''), \quad (4.41)$$

$$\nu(S(x)) = \nu(x), \quad (4.42)$$

for all $x, y \in H$.

Theorem 4.18. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then $H = \mathbf{coend}(\mathcal{C}, \omega)$ is a coribbon WHA $(H, \mu, \eta, \Delta, \varepsilon, S, r, \nu)$ where $\nu: H \rightarrow k$ is determined from the universal property of the coend by commutativity of

$$\begin{array}{ccc}
 \omega(X) & \xrightarrow{\delta_X^\omega} & \omega(X) \otimes H \\
 \omega(\nu_X) \downarrow & \searrow & \downarrow \text{id}_{\omega(X)} \otimes \nu \\
 \omega(X) & \xrightarrow{\rho_{\omega(X)}^{-1}} & \omega(X) \otimes k
 \end{array} \quad (4.43)$$

and its convolution inverse $\overline{\nu}: H \rightarrow k$ by commutativity of

$$\begin{array}{ccc}
 \omega(X) & \xrightarrow{\delta_X^\omega} & \omega(X) \otimes H \\
 \omega(\nu_X^{-1}) \downarrow & \searrow & \downarrow \text{id}_{\omega(X)} \otimes \overline{\nu} \\
 \omega(X) & \xrightarrow{\rho_{\omega(X)}^{-1}} & \omega(X) \otimes k
 \end{array} \quad (4.44)$$

for all $X \in |\mathcal{C}|$.

Lemma 4.19. Under the conditions of Theorem 4.18, the universal ribbon twist ν and its convolution inverse $\bar{\nu}$ are given by

$$\nu([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes ((\text{id}_{\widehat{\mathcal{V}}} \otimes \nu_X) \circ v)), \quad (4.45)$$

$$\bar{\nu}([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes ((\text{id}_{\widehat{\mathcal{V}}} \otimes \nu_X^{-1}) \circ v)), \quad (4.46)$$

for all $v \in \omega(X)$, $\vartheta \in \omega(X)^*$, $X \in |\mathcal{C}|$.

Proof. We have to show that the expressions (4.45) and (4.46) make the diagrams (4.43) and (4.44) commute. This follows immediately from the definitions. \square

Proof of Theorem 4.18. We verify in a direct computation that (4.45) and (4.46) satisfy the conditions of Definition 4.17. In order to see that ν and $\bar{\nu}$ are convolution inverse to each other, one just needs the dual bases of Proposition 3.7. Showing that ν is dual central requires in addition the relations (4.1). Verification of (4.41) requires all these and Proposition 3.12 as well as the condition (A.22). Finally, in order to verify (4.42), we use (4.18), the left autonomous structure of \mathcal{C} , the condition (A.23) as well as the cyclic property of the trace involved in $\text{ev}_{\omega(-)}$. \square

4.6 Special properties of the coend

Proposition 4.20. Let $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor and $H = \mathbf{coend}(\mathcal{C}, \omega)$ be the reconstructed WHA. Then

$$H_t = \{ [\vartheta|\rho_{\widehat{\mathcal{V}}}^{-1}]_{\mathbb{1}} \mid \vartheta: \widehat{\mathcal{V}} \otimes \mathbb{1} \rightarrow \widehat{\mathcal{V}} \}, \quad (4.47)$$

$$H_s = \{ [\rho_{\widehat{\mathcal{V}}}|v]_{\mathbb{1}} \mid v: \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}} \otimes \mathbb{1} \}, \quad (4.48)$$

and

$$H_t \cap H_s \cong k. \quad (4.49)$$

Proof. Let us show (4.47). For each $\vartheta \in \omega(X)^*$ and $v \in \omega(X)$, $X \in |\mathcal{C}|$, we have $\varepsilon_t([\vartheta|v]_X) = [\Phi_X(v) \circ \Psi_X(\vartheta) \circ \rho_{\widehat{\mathcal{V}}}^{-1}]_{\mathbb{1}}$, i.e. H_t is included in the set given. Conversely, for each $\vartheta \in \omega(\mathbb{1})^*$, $\varepsilon_t([\vartheta|\rho_{\widehat{\mathcal{V}}}^{-1}]_{\mathbb{1}}) = [\Phi_{\mathbb{1}}(\rho_{\widehat{\mathcal{V}}}^{-1}) \circ \Psi_{\mathbb{1}}(\vartheta) \circ \rho_{\widehat{\mathcal{V}}}^{-1}]_{\mathbb{1}} = [\vartheta|\rho_{\widehat{\mathcal{V}}}^{-1}]_{\mathbb{1}}$, using the triangle axiom of \mathcal{C} , and so the given set is contained in H_t . Let us show (4.48). For each $\vartheta \in \omega(X)^*$ and $v \in \omega(X)$, $X \in |\mathcal{C}|$, we have $\varepsilon_s([\vartheta|v]_X) = [\rho_{\widehat{\mathcal{V}}}|\rho_{\widehat{\mathcal{V}}}^{-1} \circ \eta \circ v]_{\mathbb{1}}$, i.e. H_s is included in the set given. Conversely, for each $v \in \omega(\mathbb{1})$, indeed $\varepsilon_s([\rho_{\widehat{\mathcal{V}}}|v]_{\mathbb{1}}) = [\rho_{\widehat{\mathcal{V}}}|v]_{\mathbb{1}}$, i.e. the given set is contained in H_s . Finally, (4.49) follows from (4.47) and (4.48). \square

There is one more condition that is satisfied by every WHA $H = \mathbf{coend}(\mathcal{C}, \omega)$ for a modular category \mathcal{C} and the long forgetful functor ω . This condition is the invertibility of the S -matrix. Since this condition is preserved by equivalences of semisimple k -linear ribbon categories, we take care of this in Section 6 where we show that \mathcal{C} is equivalent to the category \mathcal{M}^H of finite-dimensional right H -comodules.

Most proofs in this section were done by (1) defining the structure maps in terms of a universal property; (2) expressing these maps in terms of a convenient basis; (3) verifying their properties using this basis. Alternatively, it would have been possible to establish their properties directly from their defining commutative diagrams. In order to write down or even sketch these proofs, however, one needs extra large paper, and so we have reverted to the first method involving a basis.

5 Corepresentation theory

In this section, we consider the category \mathcal{M}^H of finite-dimensional right H -comodules of some WBA H and show that it has the structure of a monoidal category. If H is a WHA, then each object has a specified left-dual, i.e. \mathcal{M}^H is left-autonomous. If H is coquasitriangular, then \mathcal{M}^H is braided, and if H is coribbon, then \mathcal{M}^H is ribbon.

For easier reference, we have collected all definitions relevant to monoidal categories in Appendix A.1, to left-autonomous categories in Appendix A.2, to ribbon categories in Appendix A.3, and to additive and abelian categories in Appendix A.4.

5.1 Preparation

The proofs of the propositions in the section on corepresentations are all elementary although some of them are rather laborious. They rely on the following facts about WBAs, WHAs, and coquasitriangular or ribbon WHAs that we collect in this subsection. Some of them are quite challenging to verify.

Lemma 5.1. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a WBA. Then

$$\varepsilon_s(1') \otimes 1'' = 1' \otimes 1'', \quad (5.1)$$

$$\varepsilon_s(xy) = \varepsilon_s(\varepsilon_s(x)y), \quad (5.2)$$

$$h' \otimes h'' = \varepsilon_s(h') \otimes h'', \quad (5.3)$$

$$x\varepsilon_t(y) = \varepsilon(x'y''), \quad (5.4)$$

$$\varepsilon_s(x)y = y'\varepsilon(xy''), \quad (5.5)$$

$$1' \otimes (1''h) = h' \otimes h'', \quad (5.6)$$

$$x'\varepsilon_s(x'') = x, \quad (5.7)$$

$$x'\varepsilon(x''h) = xh, \quad (5.8)$$

$$(\varepsilon_s(x''))' \otimes (x'(\varepsilon_s(x''))'') = \varepsilon_s(x') \otimes x'', \quad (5.9)$$

$$\varepsilon(xy')\varepsilon_s(y'') = \varepsilon_s(xy), \quad (5.10)$$

$$(\ell h)' \otimes (\ell h)'' = \ell 1' \otimes h 1'', \quad (5.11)$$

$$(\ell h)' \otimes (\ell h)'' = 1'\ell \otimes 1''h, \quad (5.12)$$

for all $x, y \in H$ and $h \in H_s, \ell \in H_t$. If $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a WHA, then

$$\varepsilon_s(x'') \otimes (S(x')x''') = (\varepsilon_s(x))' \otimes (\varepsilon_s(x))'', \quad (5.13)$$

$$\varepsilon(hy'')y'S(y''') = h\varepsilon_t(y), \quad (5.14)$$

for all $x, y \in H$ and $h \in H_s$.

Lemma 5.2. Let H be a WBA and V be a right H -comodule. Then

$$\varepsilon(h(v_V)_H)(v_V)_V \otimes \varepsilon_s(V_H) = \varepsilon(hv_H)(v_V)_V \otimes \varepsilon_s((v_V)_H) \quad (5.15)$$

for all $h \in H_s$ and $v \in V$.

Proof. From the comodule axioms, (5.10) and (5.5). □

Lemma 5.3. Let $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ be a coquasitriangular WHA. Then

$$r(x' \otimes y') \varepsilon_t(y'') \varepsilon_s(x'') = \varepsilon_t(x') \varepsilon_s(y') r(x'' \otimes y'') \quad (5.16)$$

for all $x, y \in H$.

Proof. From the axioms, (5.11) and (5.12). \square

Lemma 5.4. Let $(H, \mu, \eta, \Delta, \varepsilon, S, r, \nu)$ be a coribbon WHA and v be its dual Drinfel'd element (4.38). Then the *pivotal form* $w: H \rightarrow k, x \mapsto v(x') \nu(x'')$ is dual group-like and satisfies

$$S^2(x) = \overline{w}(x') x'' w(x''') \quad (5.17)$$

for all $x \in H$.

Proof. Analogous to the situation in a ribbon Hopf algebra. Rather tedious. \square

5.2 Monoidal structure

The definitions in this section follow Nill [18], but are here given in a form that does not assume finite-dimensionality of H .

Proposition 5.5. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a WBA. Then H_s forms a right H -comodule with

$$\beta_{H_s}: H_s \rightarrow H_s \otimes H, \quad x \mapsto x' \otimes x''. \quad (5.18)$$

Proof. By (2.10), coassociativity and the counit property. \square

Proposition 5.6. Let H be a WBA and $V, W \in |\mathcal{M}^H|$. Then the k -vector space

$$V \widehat{\otimes} W := \{ v \otimes w \in V \otimes W \mid v \otimes w = (v_V \otimes w_W) \varepsilon(v_H w_H) \} \quad (5.19)$$

forms a right H -comodule with

$$\beta_{V \widehat{\otimes} W}: V \widehat{\otimes} W \rightarrow (V \widehat{\otimes} W) \otimes H, \quad v \otimes w \mapsto (v_V \otimes w_W) \otimes (v_H w_H). \quad (5.20)$$

Proof. Consequence of the WBA axioms and of the comodule axioms. \square

The tensor product $\widehat{\otimes}$ is often called the *truncated tensor product*. We note that the k -linear map

$$P_{V,W}: V \otimes W \rightarrow V \otimes W, \quad v \otimes w \mapsto (v_V \otimes w_W) \varepsilon(v_H w_H) \quad (5.21)$$

forms an idempotent, and that $V \widehat{\otimes} W$ is its image.

Theorem 5.7. Let H be a WBA. Then the category \mathcal{M}^H forms a k -linear monoidal category $(\mathcal{M}^H, \widehat{\otimes}, H_s, \alpha, \lambda, \rho)$ with

$$\lambda_V: H_s \widehat{\otimes} V \rightarrow V, \quad h \otimes v \mapsto v_V \varepsilon(h v_H), \quad (5.22)$$

$$\rho_V: V \widehat{\otimes} H_s \rightarrow V, \quad v \otimes h \mapsto v_V \varepsilon(v_H h), \quad (5.23)$$

and isomorphisms $\alpha_{U,V,W}: (U \widehat{\otimes} V) \widehat{\otimes} W \rightarrow U \widehat{\otimes} (V \widehat{\otimes} W)$ induced from the associator of \mathbf{Vect}_k .

Proof. (1) We have already seen that \mathcal{M}^H is k -linear as a category. Since $\widehat{\otimes}$ is k -bilinear on morphisms, \mathcal{M}^H is also k -linear as a monoidal category (c.f. Definition A.18).

(2) We claim that λ_V and ρ_V are invertible with inverses

$$\lambda_V^{-1}: V \rightarrow H_s \widehat{\otimes} V, \quad v \mapsto (1' \otimes v_V) \varepsilon(1'' v_H), \quad (5.24)$$

$$\rho_V^{-1}: V \rightarrow V \widehat{\otimes} H_s, \quad v \mapsto v_V \otimes \varepsilon_s(v_H). \quad (5.25)$$

While λ_V and ρ_V are obviously well defined, we have to verify that λ_V^{-1} maps into $H_s \widehat{\otimes} V$ which follows from (5.1) and that ρ_V^{-1} maps into $V \widehat{\otimes} H_s$ which follows from (5.9). In order to verify that $\lambda_V \circ \lambda_V^{-1} = \text{id}_V$, one needs the coaction of H on H_s , the axioms of the comodule V and the axioms of a WBA. For $\lambda_V^{-1} \circ \lambda_V = \text{id}_{H_s \widehat{\otimes} V}$, one needs in addition (5.5), (5.3) and (5.6); for $\rho_V \circ \rho_V^{-1} = \text{id}_V$, one needs (5.7) and for $\rho_V^{-1} \circ \rho_V = \text{id}_{V \widehat{\otimes} H_s}$ (5.8), (5.2) and (5.5).

(3) Using (5.3) and (5.5), one shows that λ_V is a morphism of right H -comodules and using (5.9) that ρ_V^{-1} is.

(4) Naturality of λ_V and ρ_V follows from the comodule axioms of V and from the properties of a morphism of comodules.

(5) The pentagon axiom can be proven from the comodule axioms and the axioms of a WBA, and the triangle axiom from (5.8) and (5.5). \square

Finally, we show that the forgetful functor $\mathcal{M}^H \rightarrow \mathbf{Vect}_k$ has a separable Frobenius structure (Definition 3.13). This result is precisely dual to that of [25]. First, we recall the image factorization of an idempotent in an abelian category.

Proposition 5.8 (see, for example [35]). Let \mathcal{C} be an abelian category and $p: A \rightarrow A$ be an idempotent. The image factorization of p yields an object $p(A)$ (the *image* of p), which is unique up to isomorphism, together with morphisms $\text{coim } p: A \rightarrow p(A)$ (the *coimage* map) and $\text{im } p: p(A) \rightarrow A$ (the *image* map) such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\text{coim } p} & p(A) \\ & \searrow p & \downarrow \text{im } p \\ & & A \end{array} \quad (5.26)$$

Since \mathcal{C} is abelian, the idempotent p splits. The splitting is given precisely by the two morphisms of the image factorization, and so we have $\text{id}_{p(A)} = \text{coim } p \circ \text{im } p$.

Proposition 5.9. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a WBA and $U: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ be the forgetful functor that assigns to each finite-dimensional right H -comodule X its underlying vector space UX and to each morphism of right H -comodules $f: X \rightarrow Y$ the underlying linear map $Uf: UX \rightarrow UY$. Then $(U, U_{X,Y}, U_0, U^{X,Y}, U^0)$ is a k -linear faithful functor taking values in \mathbf{fdVect}_k with a separable Frobenius structure where

$$U_{X,Y} = \text{coim } P_{X,Y}: UX \otimes UY \rightarrow P_{X,Y}(UX \otimes UY), \quad (5.27)$$

$$U_0 = \eta: k \rightarrow H_s, \quad (5.28)$$

$$U^{X,Y} = \text{im } P_{X,Y}: P_{X,Y}(UX \otimes UY) \rightarrow UX \otimes UY, \quad (5.29)$$

$$U^0 = \varepsilon|_{H_s}: H_s \rightarrow k. \quad (5.30)$$

Here, $P_{X,Y}$ is the idempotent (5.21), and so its image is the truncated tensor product, i.e. the vector space underlying the tensor product in \mathcal{M}^H ,

$$P_{X,Y}(UX \otimes UY) = U(X \widehat{\otimes} Y). \quad (5.31)$$

Furthermore, $H_s = U\mathbb{1}$ is the vector space underlying the monoidal unit (Proposition 5.5).

Proof. (1) U is k -linear and faithful because UX and Uf are just the underlying vector space and linear map, respectively.

(2) In order to see that $(U, U_{X,Y}, U_0)$ is lax monoidal (Definition A.2), we have to verify the following.

- (a) The hexagon axiom $U\alpha_{X,Y,Z} \circ U_{X \otimes Y, Z} \circ (U_{X,Y} \otimes \text{id}_{UZ}) = U_{X,Y \otimes Z} \circ (\text{id}_{UX} \otimes U_{Y,Z}) \circ \alpha_{UX,UY,UZ}$ follows from the definitions $U_{X,Y}(x \otimes y) = (x_X \otimes y_Y)\varepsilon(x_H y_H)$ and $U_{X \otimes Y, Z}(x \otimes y \otimes z) = (x_X \otimes y_Y \otimes z_Z)\varepsilon((x_H y_H)z_H)$, etc., and from the axioms of a WBA.
- (b) The first square $U\lambda_X \circ U_{\mathbb{1},X} \circ (U_0 \otimes \text{id}_{UX}) = \lambda_{UX}$ follows from the definitions $U_0(1) = 1 \in H_s$; $U_{\mathbb{1},X}(h \otimes x) = (h' \otimes x_X)\varepsilon(h''x_H)$ for $h \in H_s$, $x \in UX$; $U\lambda_X(h \otimes x) = x_X\varepsilon(hx_H)$ for $h \in H_s$, $x \in UX$; and from the axioms of a WBA. Recall that λ_{UX} on the right hand side is the unit constraint of \mathbf{Vect}_k .
- (c) The second square $U\rho_X \circ U_{X,\mathbb{1}} \circ (\text{id}_{UX} \otimes U_0) = \rho_{UX}$ follows from the definitions $U_0(1) = 1 \in H_s$; $U_{X,\mathbb{1}}(x \otimes h) = (x_X \otimes h')\varepsilon(x_H h'')$ for $x \in UX$, $h \in H_s$; $U\rho_X(x \otimes h) = x_X\varepsilon(x_H h)$ for $x \in UX$, $h \in H_s$; and from the axioms of a WBA. Again, ρ_{UX} is the unit constraint of \mathbf{Vect}_k .

(3) In order to see that $(U, U^{X,Y}, U^0)$ is oplax monoidal, we need to verify:

- (a) The hexagon axiom (A.6) holds because $U^{X,Y}(x \otimes y) = x \otimes y$ for all $x \otimes y \in U(X \widehat{\otimes} Y) \subseteq UX \otimes UY$.
- (b) In order to verify the first square $\lambda_{UX} \circ (U^0 \otimes \text{id}_{UX}) \circ U^{\mathbb{1},X}(h \otimes x) = U\lambda_X(h \otimes x)$ for all $h \otimes x \in U(\mathbb{1} \widehat{\otimes} X)$, we use the fact that $h \otimes x = (h' \otimes x_X)\varepsilon(h''x_H)$ and the definitions $U^{\mathbb{1},X}(h \otimes x) = h \otimes x$, $U^0(h) = \varepsilon(h)$, $\lambda_{UX}(1 \otimes x) = x$ (unit constraint of \mathbf{Vect}_k) and $U\lambda_X(h \otimes x) = x_X\varepsilon(hx_H)$.
- (c) In order to verify the second square $\rho_{UX} \circ (\text{id}_{UX} \otimes U^0) \circ U^{X,\mathbb{1}}(x \otimes h) = U\rho_X(x \otimes h)$ for all $x \otimes h \in U(X \widehat{\otimes} \mathbb{1})$, we use the fact that $x \otimes h = (x_X \otimes h')\varepsilon(x_H h'')$ and the definitions $U^{X,\mathbb{1}}(x \otimes h) = x \otimes h$, $U^0(h) = \varepsilon(h)$, $\rho_{UX}(x \otimes 1) = x$ and $U\rho_X(x \otimes h) = x_X\varepsilon(x_H h)$.

(4) Finally, we verify the compatibility conditions of Definition 3.13:

- (a) The splitting of the idempotent $P_{X,Y}$ yields $U_{X,Y} \circ U^{X,Y} = \text{id}_{U(X \widehat{\otimes} Y)}$.
- (b) In order to show $(\text{id}_{UX} \otimes U_{Y,Z}) \circ \alpha_{UX,UY,UZ} \circ (U^{X,Y} \otimes \text{id}_{UZ})(x \otimes y \otimes z) = U^{X,Y \otimes Z} \circ U\alpha_{X,Y,Z} \circ U_{X \otimes Y, Z}(x \otimes y \otimes z)$ for all $x \otimes y \otimes z \in U(X \widehat{\otimes} Y) \otimes UZ$, we use the fact that $x \otimes y \otimes z = (x_X \otimes y_Y \otimes z_Z)\varepsilon(x_H y_H z_H)$ as well as the axioms of a WBA.
- (c) The proof of $(U_{X,Y} \otimes \text{id}_{UZ}) \circ \alpha_{UX,UY,UZ}^{-1} \circ (\text{id}_{UX} \otimes U^{Y,Z}) = U^{X \otimes Y, Z} \circ U\alpha_{X,Y,Z}^{-1} \circ U_{X,Y \otimes Z}$ is analogous.

□

5.3 Duality

Proposition 5.10. Let H be a WHA and (V, β_V) be a finite-dimensional right H -comodule. Then the dual vector space V^* forms a right H -comodule with

$$\beta_{V^*}: V^* \rightarrow V^* \otimes H, \quad \vartheta \mapsto (v \mapsto \vartheta(v_V) \otimes S(v_H)). \quad (5.32)$$

Proof. Consequence of the WBA axioms and of the comodule axioms. \square

Theorem 5.11. Let H be a WHA. Then the category \mathcal{M}^H is left-autonomous if the left-dual of each $V \in |\mathcal{M}^H|$ is chosen as $(V^*, \text{ev}_V, \text{coev}_V)$ where V^* is the vector space dual to V and

$$\text{ev}_V: V^* \widehat{\otimes} V \rightarrow H_s, \quad \vartheta \otimes v \mapsto \vartheta(v_V) \varepsilon_s(v_H), \quad (5.33)$$

$$\text{coev}_V: H_s \rightarrow V \widehat{\otimes} V^*, \quad x \mapsto \sum_j ((e_j)_V \otimes e^j) \varepsilon(x(e_j)_H). \quad (5.34)$$

Here we have used the evaluation and coevaluation maps that turn V^* into a left-dual of V in \mathbf{Vect}_k :

$$\text{ev}_V^{(\mathbf{Vect}_k)}: V^* \otimes V \rightarrow k, \quad \vartheta \otimes v \mapsto \vartheta(v), \quad (5.35)$$

$$\text{coev}_V^{(\mathbf{Vect}_k)}: k \rightarrow V \otimes V^*, \quad 1 \mapsto \sum_j e_j \otimes e^j. \quad (5.36)$$

Proof. While ev_V is obviously well defined, we have to show that coev_V maps into $V \widehat{\otimes} V^*$. This follows from (5.1) and (5.14). In order to show that ev_V is a morphism of right H -comodules, one needs (5.13) and for coev_V one needs (5.1), (5.14), (5.4) and (5.3). The triangle identities can be proven using (5.15). \square

5.4 Ribbon structure

In this section, we show that if H is coribbon, the category \mathcal{M}^H is ribbon. As soon as we give the braiding and the ribbon twist, the proofs are straightforward.

Proposition 5.12. Let $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ be a coquasitriangular WHA. Then \mathcal{M}^H is a braided monoidal category with braiding $\sigma_{V,W}: V \widehat{\otimes} W \rightarrow W \widehat{\otimes} V$ given by

$$\sigma_{V,W}(v \otimes w) = (w_W \otimes v_V) r(w_H \otimes v_H) \quad (5.37)$$

for all $V, W \in |\mathcal{M}^H|$ and $v \in V, w \in W$. If H is cotriangular, then \mathcal{M}^H is symmetric monoidal.

Note that $Q_{V,W} = \sigma_{W,V} \circ \sigma_{V,W}$ can be obtained by $Q_{V,W}(v \otimes w) = (v_V \otimes w_W) q(v_H \otimes w_H)$, c.f. Definition 4.15.

Proposition 5.13. Let $(H, \mu, \eta, \Delta, \varepsilon, S, r, \nu)$ be a coribbon WHA. Then \mathcal{M}^H is a ribbon category with ribbon twist $\nu_V: V \rightarrow V$ given by

$$\nu_V(v) = v_V \nu(v_H) \quad (5.38)$$

for all $V \in |\mathcal{M}^H|$ and $v \in V$.

Remark 5.14. Note that the forgetful functor $U: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ of Proposition 5.9 is in general neither braided nor ribbon although both \mathcal{M}^H and \mathbf{Vect}_k are ribbon categories.

5.5 Special properties of modular categories

Proposition 5.15. Let (C, Δ, ε) be a split cosemisimple coalgebra over a field k . Then \mathcal{M}^C is semisimple according to Definition A.20. If C is in addition finite-dimensional over k , then \mathcal{M}^C is finitely semisimple.

Proof. Let $\{V_j\}_{j \in I}$ be a family of objects $V_j \in |\mathcal{M}^C|$ that contains one and only one representative per isomorphism class of irreducible right C -comodules. We show that the conditions (3a) to (3c) of Definition A.20 are satisfied.

(3a) Each V_j , $j \in I$, satisfies $\text{End}(V_j) \cong k$ by Lemma 2.15(3).

(3b) By Lemma 2.15(2).

(3c) The morphisms $\iota_\ell^{(X)}$ and $\pi_\ell^{(X)}$ are those that define the finite biproduct (2.17).

□

Lemma 5.16 (see [18, Lemma 4.5]). Let H be a WBA. If $H_t \cap H_s \cong k$, then $\text{End}(H_s) \cong k$ in \mathcal{M}^H where H_s is the monoidal unit object. In particular, every morphism $f: H_s \rightarrow H_s$ in \mathcal{M}^H is of the form $f = c \text{id}_{H_s}$ where $c \in k$ can be determined from the condition $f(1) = c1$.

Corollary 5.17. Let H be a split cosemisimple WBA such that $H_t \cap H_s \cong k$. Then \mathcal{M}^H is semisimple, and there exists a $0 \in I$ such that $H_s \cong V_0$.

Proof. By the lemma, H_s is simple in \mathcal{M}^H . By Corollary A.23, this implies that $H_s \cong V_0$ for some $0 \in I$. □

Proposition 5.18. Let H be a split cosemisimple WHA. Then \mathcal{M}^H is semisimple, and for each $j \in I$, there is some $j^* \in I$ such that $(V_j)^* \cong V_{j^*}$.

Proof. For $j \in I$, V_j is an irreducible right H -comodule. We show that every morphism $f: V_j^* \rightarrow V_j^*$ is of the form $f = c \cdot \text{id}_{V_j^*}$ with some $c \in k$, and so V_j^* is simple. By Corollary A.23, this implies that $V_j^* \cong V_{j^*}$ for some $j^* \in I$. This is done as follows. Given any $f: V_j^* \rightarrow V_j^*$, define $g: V_j \rightarrow V_j$ by

$$g = \rho_{V_j} \circ (\text{id}_{V_j} \otimes \text{ev}_{V_j}) \circ \alpha_{V_j, V_j^*, V_j} \circ ((\text{id}_{V_j} \otimes f) \otimes \text{id}_{V_j}) \circ (\text{coev}_{V_j} \otimes \text{id}_{V_j}) \circ \lambda_{V_j}^{-1}. \quad (5.39)$$

By the triangle identities,

$$f = g^* = \lambda_{V_j^*} \circ (\text{ev}_{V_j} \otimes \text{id}_{V_j^*}) \circ \alpha_{V_j^*, V_j, V_j^*}^{-1} \circ (\text{id}_{V_j^*} \otimes (g \otimes \text{id}_{V_j^*})) \circ (\text{id}_{V_j^*} \otimes \text{coev}_{V_j}) \circ \rho_{V_j^*}^{-1}, \quad (5.40)$$

but since V_j is simple by assumption, $g = c \cdot \text{id}_{V_j}$ for some $c \in k$, and we can use k -linearity of \mathcal{M}^H as a monoidal category and another triangle identity in order to show that $f = c \cdot \text{id}_{V_j^*}$. □

Lemma 5.19. Let H be a coribbon WHA, $V \in |\mathcal{M}^H|$ and $f: V \rightarrow V$. Then the trace $\text{tr}_V(f): H_s \rightarrow H_s$ is given by

$$\text{tr}_V(f)(h) = \sum_{j, \ell=1}^n f_{j\ell} \varepsilon_s(S((hc_{\ell j})')) w((hc_{\ell j})'') \quad (5.41)$$

for all $h \in H_s$. Here $n = \dim_k(V)$ is the k -dimension of V ; the $f_{j\ell} \in k$ are the matrix elements of f , i.e. $f(v_\ell) = \sum_{j=1}^n v_j f_{j\ell}$; the $c_{\ell j} \in H$ are the coefficients of V , i.e. $\beta_V(v_j) = \sum_{\ell=1}^n v_\ell \otimes c_{\ell j}$; and $w: H \rightarrow k$ is the pivotal form (Lemma 5.4).

Proof. The trace of f in the ribbon category \mathcal{M}^H is given by

$$\mathrm{tr}_V(f) = \mathrm{ev}_V \circ \sigma_{V,V^*} \circ (\nu_V \otimes \mathrm{id}_{V^*}) \circ (f \otimes \mathrm{id}_{V^*}) \circ \mathrm{coev}_V: H_s \rightarrow H_s. \quad (5.42)$$

We insert the definitions of these morphisms and use (5.5) and (5.16). \square

In the following, we restrict ourselves to coribbon WHAs in which $H_t \cap H_s \cong k$. By Lemma 5.16, this implies that $\mathrm{End}(H_s) \cong k$, and so all traces take values in the field k . In order to express the coefficients of the S -matrix (2.2) of \mathcal{M}^H in terms of the linear form $q: H \otimes H \rightarrow k$ of (4.36), we proceed as follows.

Definition 5.20. Let H be a coribbon WHA over a field k such that $H_t \cap H_s \cong k$ and $V \in |\mathcal{M}^H|$, $n = \dim_k V$. We call

$$\chi_V = \sum_{j=1}^n c_{jj} \in H \quad (5.43)$$

the *dual character* of V and

$$T_V = \sum_{j,\ell=1}^n c_{j\ell} w(c_{\ell j}) \in H \quad (5.44)$$

the *dual quantum character* of V .

Note that a dual central linear form $\alpha: H \rightarrow k$ defines a natural transformation of the identity functor $f^{(\alpha)}: 1_{\mathcal{M}^H} \Rightarrow 1_{\mathcal{M}^H}$ via $f_V^{(\alpha)} = (\mathrm{id}_V \otimes \alpha) \circ \beta_V$ for all $V \in |\mathcal{M}^H|$. Traces in \mathcal{M}^H can thus be expressed in terms of the dual quantum characters.

Proposition 5.21. Let H be a coribbon WHA over a field k such that $H_t \cap H_s \cong k$, $V \in |\mathcal{M}^H|$, and $\alpha: H \rightarrow k$ be dual central. Then

$$\mathrm{tr}_V(f_V^{(\alpha)}) = c_V^{(\alpha)} \mathrm{id}_{H_s}, \quad (5.45)$$

where the element $c_V^{(\alpha)} \in k$ is determined by

$$\alpha(T'_V) \varepsilon_s(S(T''_V)) = c_V^{(\alpha)} \eta(1). \quad (5.46)$$

Proof. By Lemma 5.16, we can determine $c_V^{(\alpha)}$ in (5.45) by evaluation at $\eta(1) \in H_s$. We use the formula (5.41). \square

In the special case in which H is a Hopf algebra, we have $H_s \cong k$ and $\varepsilon_s = \varepsilon$, and so Proposition 5.21 reduces to $\mathrm{tr}_V(f_V^{(\alpha)}) = \alpha(T_V)$ as expected. In the case in which H is a WHA and $\varepsilon(\eta(1)) \neq 0$, we can apply ε to (5.46) and obtain $c_V^{(\alpha)} = \alpha(T_V)/\varepsilon(\eta(1))$.

In order to deal with the S -matrix, we need the analogue of Proposition 5.21 for endomorphisms of a tensor product of modules. Note that a linear form $\gamma: H \otimes H \rightarrow k$ that satisfies

$$x'y'\gamma(x'' \otimes y'') = \gamma(x' \otimes y')x''y'', \quad (5.47)$$

$$\varepsilon(x'y')\gamma(x'' \otimes y'') = \gamma(x \otimes y), \quad (5.48)$$

for all $x, y \in H$, defines a morphism $f_{V,W}^{(\gamma)} \in \mathrm{End}(V \widehat{\otimes} W)$ via

$$f_{V,W}^{(\gamma)} = (\mathrm{id}_{V \widehat{\otimes} W} \otimes \gamma) \circ (\mathrm{id}_V \otimes \sigma_{H,W} \otimes \mathrm{id}_H) \circ (\beta_V \otimes \beta_W) \quad (5.49)$$

In particular, $q: H \otimes H \rightarrow k$ of (4.36) satisfies these conditions and gives rise to the morphism $f_{V,W}^{(q)} = Q_{V,W} = \sigma_{W,V} \circ \sigma_{V,W}$ whose trace in \mathcal{M}^H is the S -matrix.

Proposition 5.22. Let H be a coribbon WHA over a field k such that $H_t \cap H_s \cong k$, $V, W \in |\mathcal{M}^H|$ and $\gamma: H \otimes H \rightarrow k$ be a linear form that satisfies (5.47) and (5.48). Then

$$\mathrm{tr}_{V \widehat{\otimes} W}(f_{V,W}^{(\gamma)}) = c_{V,W}^{(\gamma)} \mathrm{id}_{H_s} \quad (5.50)$$

where the element $c_{V,W}^{(\gamma)} \in k$ is determined by

$$\gamma(T'_V \otimes T'_W) \varepsilon_s(S(T''_V T''_W)) = c_{V,W}^{(\gamma)} \eta(1). \quad (5.51)$$

Proof. Analogous to Proposition 5.21. Apply (5.41) to $V \widehat{\otimes} W$ and exploit that $w: H \rightarrow k$ is dual group-like. \square

In the special case in which H is a Hopf algebra, Proposition 5.22 reduces to $\mathrm{tr}_{V \widehat{\otimes} W}(f_{V,W}^{(\gamma)}) = \gamma(T_V \otimes T_W)$ as expected. In the case in which H is a WHA and $\varepsilon(\eta(1)) \neq 0$, we can apply ε to (5.51) and obtain $c_{V,W}^{(\gamma)} = \gamma(T_V \otimes T_W) / \varepsilon(\eta(1))$.

Definition 5.23. Let H be a coribbon WHA over a field k such that $H_t \cap H_s \cong k$. Let $T(H) = \mathrm{span}_k\{T_V \mid V \in |\mathcal{M}^H|\} \subseteq H$ denote the space of dual quantum characters of H . We define a linear form $\tilde{q}: T(H) \otimes T(H) \rightarrow k$, $T_V \otimes T_W \rightarrow \tilde{q}_{V,W}$ where the $\tilde{q}_{V,W} \in k$ are determined by

$$q(T'_V \otimes T'_W) \varepsilon_s(S(T''_V T''_W)) = \tilde{q}_{V,W} \eta(1). \quad (5.52)$$

H is called *weakly cofactorizable* if every linear form $\varphi: T(H) \rightarrow k$ is of the form $\varphi(-) = \tilde{q}(- \otimes x)$ for some $x \in T(H)$.

In the special cases in which H is a Hopf algebra or in which H is a WHA with $\varepsilon(\eta(1)) \neq 0$, the condition of weak cofactorizability reduces to the requirement that the bilinear form $q: H \otimes H \rightarrow k$ be non-degenerate if restricted to $T(H)$. The following Corollary finally spells out the relationship with the S -matrix of \mathcal{M}^H .

Corollary 5.24. Let H be a finite-dimensional, split cosemisimple, coribbon WHA over a field k such that $H_t \cap H_s \cong k$. Let $\{V_j\}_{j \in I}$ denote a set of representatives of the isomorphism classes of simple objects of \mathcal{M}^H . H is weakly cofactorizable if and only if the matrix with coefficients $S_{j\ell} = \tilde{q}_{V_j, V_\ell}$ is invertible.

The results of the present section can be summarized as follows.

Theorem 5.25. Let H be a finite-dimensional, split cosemisimple, coribbon WHA over a field k such that $H_t \cap H_s \cong k$.

- (1) \mathcal{M}^H is a finitely semisimple additive ribbon category.
- (2) \mathcal{M}^H is modular if and only if H is weakly cofactorizable.

The following terminology is therefore appropriate.

Definition 5.26. Let H be a WHA over a field k . H is called *comodular* if H is a finite-dimensional, split cosemisimple, weakly cofactorizable, coribbon WHA such that $H_t \cap H_s \cong k$.

Theorem 5.25(2) generalizes the result of Takeuchi [36, Theorem 4.6(1)] twofold: (1) from Hopf algebras to WHAs and (2) by removing the assumptions on the underlying field k .

Note that the condition of split cosemisimplicity appears as a consequence of requiring $\text{End}(V_j) \cong k$ for the simple objects of a modular category, rather than allowing $\text{End}(V_j)$ to be a finite skew field extension over k . Nevertheless, everything that has been done so far, works for any field k .

In Section 6 we show that if \mathcal{C} satisfies all conditions of Definition 2.1 except maybe for (3), i.e. non-degeneracy of the S matrix, and if $H = \mathbf{coend}(\mathcal{C}, \omega)$ is the WHA reconstructed from \mathcal{C} with respect to the long forgetful functor ω , then weak cofactorizability of H is both necessary and sufficient for the non-degeneracy of the S -matrix.

Compared with the sufficient conditions stated in [29, Lemma 8.2], we have not only removed assumptions on the underlying field k (algebraic closure and that the characteristic of k does not divide $\dim_k(H_s)$), but our condition of weak cofactorizability is indeed weaker than the condition (dual to) factorizability used in [29]. That condition would read in our context as follows.

A coquasitriangular WHA H is called *cofactorizable* if every linear form $\varphi: H \rightarrow k$ that satisfies

$$\varphi(y')\varepsilon_t(y'') = \varepsilon_t(y')\varphi(y'') \quad (5.53)$$

for all $y \in H$, is of the form $\varphi(-) = q(- \otimes x)$ for some $x \in H$. The condition of weak cofactorizability, in contrast, requires non-degeneracy of q only on dual quantum characters.

5.6 Further properties

In this section, we collect further results on the reconstructed WHA.

Definition 5.27. Let H be a WBA.

- (1) H is called *copure* if the monoidal unit object H_s of \mathcal{M}^H is irreducible.
- (2) H is called *connected* if $Z(H) \cap H_t \cong k$.
- (3) If H is finite-dimensional, H is called *coconnected* if H^* is connected.
- (4) H is called a *face algebra* [37] if H_s is a commutative algebra.

Proposition 5.28. Let \mathcal{C} be a modular category, $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor and $H = \mathbf{coend}(\mathcal{C}, \omega)$ be the reconstructed WHA.

- (1) $H_t \cong R$ are isomorphic as k -algebras.
- (2) $H_s \cong R$ are isomorphic as k -algebras.
- (3) $H_{\min} \cong R^{\text{op}} \otimes R$ are isomorphic as k -algebras and

$$H_{\min} = \omega(\mathbb{1})^* \otimes \omega(\mathbb{1}) = \{ [\vartheta|v]_{\mathbb{1}} \mid v: \widehat{V} \rightarrow \widehat{V} \otimes \mathbb{1}; \vartheta: \widehat{V} \otimes \mathbb{1} \rightarrow \widehat{V} \}. \quad (5.54)$$

- (4) H is regular.
- (5) H is a face algebra.
- (6) The dual Drinfel'd elements of H are given by

$$u([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes ((D_{\widehat{V}}^{-1} \otimes \nu_X) \circ v \circ D_{\widehat{V}})), \quad (5.55)$$

$$v([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes ((D_{\widehat{V}} \otimes \nu_X^{-1}) \circ v \circ D_{\widehat{V}}^{-1})), \quad (5.56)$$

for all $v \in \omega(X)$, $\vartheta \in \omega(X)^*$ and $X \in |\mathcal{C}|$.

(7) The pivotal form $w: H \rightarrow k$ (Lemma 5.4) is given by

$$w([\vartheta|v]_X) = \text{ev}_{\omega(X)}((D_{\widehat{V}}^{-1} \circ \vartheta \circ (D_{\widehat{V}} \otimes \text{id}_X)) \otimes v), \quad (5.57)$$

for all $v \in \omega(X)$, $\vartheta \in \omega(X)^*$ and $X \in |\mathcal{C}|$.

(8) The dual character associated with $X \in |\mathcal{C}|$ is the element

$$\sum_j [e_{(X)}^j | e_j^{(X)}]_X \in H. \quad (5.58)$$

(9) The dual quantum character associated with $X \in |\mathcal{C}|$ is the element

$$\sum_j [D_{\widehat{V}}^{-1} \circ e_{(X)}^j \circ (D_{\widehat{V}} \otimes \text{id}_X) | e_j^{(X)}]_X \in H. \quad (5.59)$$

Proof. (1) From the idempotent basis $(\lambda_j)_j$ of Proposition 3.5, one obtains a pair of dual bases $(e_j^{(1)})_j$ and $(e_{(1)}^j)_j$ of $\omega(\mathbb{1})$ and $\omega(\mathbb{1})^*$ with respect to $\text{ev}_{\omega(\mathbb{1})}$ by

$$e_j^{(1)} = \rho_{\widehat{V}}^{-1} \circ \lambda_j, \quad e_{(1)}^j = \lambda_j \circ \rho_{\widehat{V}}. \quad (5.60)$$

It follows from the triangle axiom of \mathcal{C} that these form a basis of orthogonal idempotents of H_t .

(2) Analogous.

(3) According to [21, Section 3], H_{\min} is generated as an algebra by $H_t \cup H_s$. From Proposition 4.20, we see that

$$\omega(\mathbb{1})^* \otimes \omega(\mathbb{1}) = \text{span}_k \{ [\vartheta|v]_{\mathbb{1}} \mid v: \widehat{V} \rightarrow \widehat{V} \otimes \mathbb{1}; \vartheta: \widehat{V} \otimes \mathbb{1} \rightarrow \widehat{V} \} \quad (5.61)$$

is the vector space generated by $H_t \cup H_s$. We verify in a direct calculation that it is an algebra with unit $[\rho_{\widehat{V}} | \rho_{\widehat{V}}^{-1}]_{\mathbb{1}}$ and multiplication

$$\mu([e_{(1)}^j | e_{\ell}^{(1)}]_{\mathbb{1}} \otimes [e_{(1)}^m | e_n^{(1)}]_{\mathbb{1}}) = \delta_{jm} \delta_{\ell n} [e_{(1)}^j | e_{\ell}^{(1)}]_{\mathbb{1}}, \quad (5.62)$$

using again the triangle axiom. This equation also shows the isomorphism of algebras $H_{\min} \cong R \otimes R \cong R^{\text{op}} \otimes R$.

(4) Using (4.21),

$$\begin{aligned} S^2([e_{(1)}^j | e_{\ell}^{(1)}]_{\mathbb{1}}) &= [D_{\widehat{V}}^{-1} \circ e_{(1)}^j \circ (D_{\widehat{V}} \otimes \text{id}_{\mathbb{1}}) | (D_{\widehat{V}}^{-1} \otimes \text{id}_{\mathbb{1}}) \circ e_{\ell}^{(1)} \circ D_{\widehat{V}}]_{\mathbb{1}} \\ &= [e_{(1)}^j | e_{\ell}^{(1)}]_{\mathbb{1}}, \end{aligned} \quad (5.63)$$

for all $j, \ell \in I$ by naturality of $\rho_{\widehat{V}}$.

(5) By (1).

The remaining claims are proven by direct computation. \square

Proposition 5.29. Every comodular WHA H is coconnected and copure.

Proof. Since H is finite-dimensional and $H_t \cap H_s \cong k$, H is coconnected by [38, Proposition 3.11]. By Corollary 5.17, $H_s \cong V_0$ for some $0 \in I$. But by Proposition 5.15, each V_j , $j \in I$, is an irreducible right H -comodule. \square

6 Equivalence of categories

Let us compare the original modular category \mathcal{C} with the category \mathcal{M}^H of finite-dimensional right H -comodules of the reconstructed WHA $H = \mathbf{coend}(\mathcal{C}, \omega)$. We show that $\mathcal{C} \simeq \mathcal{M}^H$ are equivalent as ribbon categories. In this section, most commutative diagrams are in \mathcal{M}^H . We highlight this fact by putting the hat on the truncated tensor product $\widehat{\otimes}$ and by writing H_s rather than $\mathbb{1}$ for its monoidal unit object.

6.1 Equivalence of monoidal categories

In order to see that $\mathcal{C} \simeq \mathcal{M}^H$ are equivalent as monoidal categories, we show the following.

Theorem 6.1. Let \mathcal{C} be a modular category, $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor and $H = \mathbf{coend}(\mathcal{C}, \omega)$ be the reconstructed WHA.

- (1) The long forgetful functor factors through \mathcal{M}^H , i.e. the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{M}^H \\ & \searrow \omega & \downarrow U \\ & & \mathbf{Vect}_k \end{array} \quad (6.1)$$

commutes. Here $U: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ is the forgetful functor of Proposition 5.9.

- (2) The functor F is k -linear, essentially surjective and fully faithful.
 (3) $(F, F_{X,Y}, F_0)$ forms a strong monoidal functor with

$$F_{X,Y}: FX \widehat{\otimes} FY \rightarrow F(X \otimes Y), \quad f \otimes g \mapsto \alpha_{\widehat{V}, X, Y} \circ (f \otimes \mathrm{id}_Y) \circ g, \quad (6.2)$$

$$F_0: H_s \mapsto F\mathbb{1}, \quad [\rho_{\widehat{V}}|v]_{\mathbb{1}} \mapsto v. \quad (6.3)$$

Proof. (1) The functor F sends the objects and morphisms of \mathcal{C} to the same vector spaces and linear maps as ω does, i.e. $FX = \omega(X)$ and $Ff = \omega(f)$ for all $X, Y \in |\mathcal{C}|$ and $f: X \rightarrow Y$. In order to show that F is well defined as a functor to \mathcal{M}^H , we have to verify the following:

- (a) For each $X \in |\mathcal{C}|$, $FX = \omega(X)$ forms a right H -comodule, c.f. (2.26).
 (b) For each morphism $f: X \rightarrow Y$ of \mathcal{C} , $Ff = \omega(f) = (\mathrm{id}_{\widehat{V}} \otimes f) \circ -$ is a morphism of right H -comodules because

$$\delta_Y^\omega \circ \omega(f)[v] = (\omega(f) \otimes \mathrm{id}_H) \circ \delta_X^\omega[v] \quad (6.4)$$

for all $v \in \omega(X)$. This can be verified by using the coaction (2.26), the relations (4.1), the form of the dual morphism (3.10) and the properties of the coevaluation (3.9).

- (2) F is obviously k -linear since ω is. F is essentially surjective because H is split cosemisimple and therefore every finite-dimensional right H -comodule M is of the form

$$M \cong \bigoplus_{\ell=1}^n \omega(V_{j_\ell}) \cong \omega \left(\bigoplus_{\ell=1}^n V_{j_\ell} \right) \quad (6.5)$$

for some $j_1, \dots, j_n \in I$. Here we have used that \mathcal{C} is additive and thus has all finite biproducts. F is faithful because ω is (Proposition 3.8). In order to see that F is full, consider some morphism $f: M \rightarrow N$ of \mathcal{M}^H , decompose both source and target as in (6.5). Since the $\omega(V_j)$, $j \in I$, are simple in \mathcal{M}^H , morphisms between these are either null or multiples of the identity. The latter are of the form $(\text{id}_{\widehat{V}} \otimes \text{id}_{V_j}) \circ - = \omega(\text{id}_{V_j})$.

- (3) In order to see that $F_{X,Y}$ and F_0 are well defined, we have to show that these linear maps are morphisms of right H -comodules, *i.e.*

- (a) For all $[\rho_{\widehat{V}}|v]_{\mathbb{1}} \in H_s$, we have

$$\delta_{\mathbb{1}}^\omega \circ F_0([\rho_{\widehat{V}}|v]_{\mathbb{1}}) = (F_0 \otimes \text{id}_H) \circ \beta_{H_s}([\rho_{\widehat{V}}|v]_{\mathbb{1}}). \quad (6.6)$$

In order to show this, we need the coaction (2.26) on $F\mathbb{1} = \omega(\mathbb{1})$ and the coaction (5.18) on H_s . Recall that the subalgebra H_s of Proposition 2.6 was computed for the reconstructed WHA in Proposition 4.20(2).

- (b) For all $X, Y \in |\mathcal{C}|$, $v \in FX = \omega(X)$ and $w \in FY = \omega(Y)$, we find

$$\begin{aligned} & (F_{X,Y} \otimes \text{id}_H) \circ (\text{id}_{\omega(X)} \otimes \text{id}_{\omega(Y)} \otimes \mu) \circ (\text{id}_{\omega(X)} \otimes \sigma_{H,\omega(Y)} \otimes \text{id}_H) \\ & \circ (\delta_X^\omega \otimes \delta_Y^\omega)[v \otimes w] \\ &= \delta_{X \otimes Y}^\omega \circ F_{X,Y}[v \otimes w], \end{aligned} \quad (6.7)$$

where the left hand side is the comodule structure of the tensor product $FX \widehat{\otimes} FY$ from (5.20). In order to verify the equation, we use the coactions of (2.26) and Proposition 3.12.

F_0 is an isomorphism with inverse

$$F_0^{-1}: F\mathbb{1} \rightarrow H_s, \quad v \mapsto [\rho_{\widehat{V}}|v]_{\mathbb{1}}. \quad (6.8)$$

In order to see that the $F_{X,Y}$ are isomorphisms, we note that $F_{X,Y}$ is the restriction of $\omega_{X,Y}$ of (3.27) to $FX \widehat{\otimes} FY$, *i.e.* the restriction to the truncated tensor product of \mathcal{M}^H , *c.f.* (5.19).

We see that $\omega^{X,Y} \circ \omega_{X,Y}: \omega(X) \otimes \omega(Y) \rightarrow \omega(X) \otimes \omega(Y)$ agrees with the idempotent $P_{\omega(X),\omega(Y)}$ of (5.21) because for all $v \in \omega(X)$ and $w \in \omega(Y)$,

$$\begin{aligned} & \omega^{X,Y} \circ \omega_{X,Y}[v \otimes w] \\ &= (\text{id}_{\omega(X)} \otimes \text{id}_{\omega(Y)} \otimes (\varepsilon \circ \mu)) \circ (\text{id}_{\omega(X)} \otimes \sigma_{H,\omega(Y)} \otimes \text{id}_H) \circ (\delta_X^\omega \otimes \delta_Y^\omega)[v \otimes w] \\ &= P_{\omega(X),\omega(Y)}(v \otimes w), \end{aligned} \quad (6.9)$$

using the coactions (2.26) and Proposition 3.12.

Therefore $\omega^{X,Y}$ maps into the truncated tensor product $FX \widehat{\otimes} FZ$. It is a left-inverse of $F_{X,Y}$ because of (6.9) and a right-inverse because of (3.23).

Finally $F_{X,Y}$ satisfies the hexagon axiom of a strong monoidal functor because $\omega_{X,Y}$ does. The two square axioms have to be verified explicitly:

- (a) First, for all $[\rho_{\widehat{V}}|v]_{\mathbb{1}} \in H_s$ and $w \in \omega(X)$, we have

$$\begin{aligned} & F\lambda_X \circ F_{\mathbb{1},X} \circ (F_0 \otimes \text{id}_{FX})([\rho_{\widehat{V}}|v]_{\mathbb{1}} \otimes w) \\ &= ((\varepsilon \circ \mu) \otimes \text{id}_{FX}) \circ (\text{id}_{H_s} \otimes \sigma_{FX,H}) \circ (\text{id}_{H_s} \otimes \delta_X^\omega)([\rho_{\widehat{V}}|v]_{\mathbb{1}} \otimes w) \\ &= \lambda_{FX}([\rho_{\widehat{V}}|v]_{\mathbb{1}} \otimes w), \end{aligned} \quad (6.10)$$

using the triangle axiom in \mathcal{C} , the relations (4.1), the left-duals of Proposition 3.7 and (5.8).

(b) Similarly, we verify

$$\begin{aligned} & F\rho_X \circ F_{X,1} \circ (\text{id}_{FX} \otimes F_0)(w \otimes [\rho_{\widehat{V}}|v]_1) \\ &= (\text{id}_{FX} \otimes (\varepsilon \circ \mu)) \circ (\delta_X^\omega \otimes \text{id}_H)(w \otimes [\rho_{\widehat{V}}|v]_1) \\ &= \rho_{FX}(w \otimes [\rho_{\widehat{V}}|v]_1). \end{aligned} \tag{6.11}$$

□

Corollary 6.2. Under the conditions of Theorem 6.1, the categories

$$\mathcal{C} \simeq_{\otimes} \mathcal{M}^{\text{coend}(\mathcal{C}, \omega)} \tag{6.12}$$

are equivalent as monoidal categories.

Proof. Since F is essentially surjective and fully faithful, by [39, Theorem IV.4.1], F is part of an adjoint equivalence $F \dashv G$. By Proposition A.4, the fact that F is strong monoidal implies that G is lax monoidal and both unit and counit of the adjoint equivalence are monoidal natural transformations. □

6.2 Equivalence of ribbon categories

In this section, we show that the original modular category \mathcal{C} is equivalent to the category \mathcal{M}^H , $H = \text{coend}(\mathcal{C}, \omega)$, as a ribbon category.

Proposition 6.3. Let \mathcal{C} be a modular category, $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor and $H = \text{coend}(\mathcal{C}, \omega)$ the reconstructed coquasitriangular WHA. Then the functor $F: \mathcal{C} \rightarrow \mathcal{M}^H$ of Theorem 6.1 is braided.

Proof. We have to show that the condition (A.21) holds for F , i.e. that

$$\sigma_{FX, FY}(v \otimes w) = F_{Y, X}^{-1} \circ F(\sigma_{X, Y}) \circ F_{X, Y}(v \otimes w) \tag{6.13}$$

for all $v \in FX = \omega(X)$, $w \in FY = \omega(Y)$, $X, Y \in |\mathcal{C}|$. Here, $\sigma_{FX, FY}$ is the braiding obtained in Proposition 5.12 from the coquasitriangular structure of Theorem 4.13. The claim is an immediate consequence of the definitions. □

Proposition 6.4. Let \mathcal{C} be a modular category, $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor and $H = \text{coend}(\mathcal{C}, \omega)$ be the reconstructed coribbon WHA. Then the functor $F: \mathcal{C} \rightarrow \mathcal{M}^H$ of Theorem 6.1 is ribbon.

Proof. We have to show that the condition (A.29) holds for F , i.e. that

$$\nu_{FX}(v) = F(\nu_X)(v) \tag{6.14}$$

for all $v \in FX = \omega(X)$, $X \in |\mathcal{C}|$. Here, ν_{FX} is the ribbon twist obtained in Proposition 5.13 from the coribbon structure of Theorem 4.18. The claim follows immediately from the definitions. □

6.3 Equivalence of modular categories

Definition 6.5. Let \mathcal{C} and \mathcal{C}' be modular categories with the same $k = \text{End}(\mathbb{1}) = \text{End}(\mathbb{1}')$. We say that \mathcal{C} and \mathcal{C}' are *equivalent as modular categories* if there is a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ that is k -linear, essentially surjective, fully faithful, strong monoidal and ribbon.

Theorem 6.6. Let \mathcal{C} be a modular category and ω be the long forgetful functor. Then

$$\mathcal{C} \simeq \mathcal{M}^{\mathbf{coend}(\mathcal{C}, \omega)} \quad (6.15)$$

are equivalent as modular categories.

Proof. The functor F of Theorem 6.1 has these properties. \square

Corollary 6.7. Each modular category \mathcal{C} is abelian, and its long forgetful functor ω is exact.

Proof. Since the functor $F: \mathcal{C} \rightarrow \mathcal{M}^{\mathbf{coend}(\mathcal{C}, \omega)}$ is part of an equivalence of categories and the category $\mathcal{M}^{\mathbf{coend}(\mathcal{C}, \omega)}$ is abelian, so is \mathcal{C} . As part of an equivalence, F is exact, and since U of Theorem 6.1 is exact, too, so is ω . \square

Finally, we can complete the characterization of the WHA $H = \mathbf{coend}(\mathcal{C}, \omega)$ reconstructed from a modular category \mathcal{C} and the long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ (Section 4). The following result complements Theorem 5.25(2).

Theorem 6.8. Let \mathcal{C} be a modular category with $k = \text{End}(\mathbb{1})$ and $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ be the long forgetful functor. Then $H = \mathbf{coend}(\mathcal{C}, \omega)$ is a comodular WHA over k .

Proof. $\mathcal{C} \simeq \mathcal{M}^H$ are equivalent as modular categories, and $F: \mathcal{C} \rightarrow \mathcal{M}^H$ of Theorem 6.1 is a k -linear, essentially surjective, fully faithful, strong monoidal ribbon functor. Such a functor preserves simple objects up to isomorphism and preserves traces (Proposition A.28), and so it preserves the non-degeneracy of the S -matrix of Definition 2.1(3) as well. Then Corollary 5.24 implies weak cofactorizability of H , and so H is comodular. \square

6.4 Morita equivalence and the choice of the forgetful functor

When we start with a modular category \mathcal{C} and reconstruct a comodular Weak Hopf Algebra $H = \mathbf{coend}(\mathcal{C}, \omega)$, we always work with the canonical choice of the long forgetful functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$. When we start with a comodular WHA H over k , however, the category \mathcal{M}^H always comes with a forgetful functor $U: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ (Proposition 5.9) which may or may not agree with the canonical choice of the long forgetful functor. In order to better understand the situation, let us recall the following results from [13, Theorem 2.1.12 and Lemma 2.2.1].

There is a category \mathfrak{C}_k whose objects (\mathcal{C}, ω) are *categories over \mathbf{Vect}_k* , i.e. pairs of a small category \mathcal{C} with a functor $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ that takes values in \mathbf{fdVect}_k . Its morphisms are *functors over \mathbf{Vect}_k* . A functor $[F, \xi]: (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ over \mathbf{Vect}_k is an equivalence class of pairs (F, ξ) where $F: \mathcal{C} \rightarrow \mathcal{C}'$ is a functor and $\xi: \omega \Rightarrow \omega' \circ F$ a natural equivalence. The equivalence relation is such that $[F, \xi]$ is an isomorphism in \mathfrak{C}_k if and only if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an equivalence of categories.

Then Tannaka–Kreĭn duality between coalgebras and their categories of finite-dimensional comodules is an adjunction

$$\begin{array}{ccc} & \text{Coend}(-) & \\ \mathfrak{C}_k & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathbf{CoAlg}_k \\ & \mathcal{M}^- & \end{array} \quad (6.16)$$

Here \mathbf{CoAlg}_k is the category of coalgebras over k and their homomorphisms. The functor **coend** applied to a category (\mathcal{C}, ω) over \mathbf{Vect}_k gives the universal coend, using the functor ω supplied, and the functor \mathcal{M}^- applied to a coalgebra H gives the category \mathcal{M}^H of finite-dimensional right H -comodules with the forgetful functor U^H of Proposition 5.9, viewed as a category (\mathcal{M}^H, U^H) over \mathbf{Vect}_k .

The counit of the adjunction is always a natural equivalence, i.e. for each coalgebra H over k , $H \cong \mathbf{coend}(\mathcal{M}^H, U)$ are isomorphic as coalgebras. If (\mathcal{C}, ω) is a category over \mathbf{Vect}_k for which \mathcal{C} is k -linear abelian and ω is k -linear, faithful and exact, then the unit of the adjunction is an isomorphism as well, i.e. (\mathcal{C}, ω) is isomorphic in \mathfrak{C}_k to $(\mathcal{M}^{\mathbf{coend}(\mathcal{C}, \omega)}, U^{\mathbf{coend}(\mathcal{C}, \omega)})$. This means that $\mathcal{C} \simeq \mathcal{M}^{\mathbf{coend}(\mathcal{C}, \omega)}$ are equivalent as categories. We have shown this by hand for the case in which \mathcal{C} is modular and ω the long forgetful functor.

If one starts with a comodular WHA H whose underlying functor $U^H: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ is not naturally isomorphic to the long forgetful functor $\omega: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$, the above adjunction yields a coalgebra $\mathbf{coend}(\mathcal{M}^H, U^H)$ that is isomorphic to H , but the coalgebra we have reconstructed in Section 4, is $\mathbf{coend}(\mathcal{M}^H, \omega)$ which need not be isomorphic to H as a coalgebra.

Our $\mathbf{coend}(\mathcal{M}^H, \omega)$ is in general only Morita equivalent to H . It is simply a canonical choice in the class of all comodular WHAs whose categories of finite-dimensional comodules are equivalent to \mathcal{M}^H as modular categories.

7 Example

In this section, we present the reconstructed Weak Hopf Algebra H for the modular category \mathcal{C} associated with the quantum group $U_q(\mathfrak{sl}_2)$, q a root of unity. We use the diagrammatic description of [40] and precisely follow their notation.

Let $r \in \{2, 3, 4, \dots\}$ and $q = \exp \frac{\pi}{r}$. For simplicity, we work over the complex numbers $k = \mathbb{C}$. The isomorphism classes of simple objects of \mathcal{C} are indexed by the set $I = \{0, 1, \dots, r-2\}$. By V_j , we denote a specific representative of the class $j \in I$. Its identity morphism is visualized by a straight line, labeled by $j \in I$,

$$\begin{array}{c} |^j \end{array} \quad (7.1)$$

All our diagrams are plane projections of oriented framed tangles, drawn in blackboard framing. The coherence theorem for ribbon categories [7] makes sure that each diagram defines a morphism of \mathcal{C} . Since \mathcal{C} is k -linear, we can take formal linear combinations of diagrams with coefficients in k . All our diagrams are read from top to bottom.

Two special features of $U_q(\mathfrak{sl}_2)$ are exploited. First, the simple objects are isomorphic to their duals, and the choice of representatives V_j , $j \in I$, of the simple objects is such that $(V_j)^* = V_j$ are equal rather than merely isomorphic. This allows us to omit any arrows from the diagrams that would indicate the orientation of the ribbon tangle.

Second, there are no higher multiplicities, i.e. for all $a, b, c \in I$, we have $\dim_k \text{Hom}(V_a \otimes V_b, V_c) \in \{0, 1\}$. More precisely, $\text{Hom}(V_a \otimes V_b, V_c) \cong k$ if and only if the triple (a, b, c) is *admissible*. Otherwise, $\text{Hom}(V_a \otimes V_b, V_c) = \{0\}$.

Definition 7.1. A triple $(a, b, c) \in I^3$ is called *admissible* if the following conditions hold.

- (1) $a + b + c \equiv 0 \pmod{2}$ (*parity*),
- (2) $a + b - c \geq 0$ and $b + c - a \geq 0$ and $c + a - b \geq 0$ (*quantum triangle inequality*),
- (3) $a + b + c \leq 2r - 4$ (*non-negligibility*).

A special choice of basis vector of $\text{Hom}(V_a \otimes V_b, V_c)$ is denoted by a trivalent vertex,

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \\ c \end{array} \quad (7.2)$$

If we draw such a diagram for a triple $(a, b, c) \in I^3$ that is not admissible, by convention, we multiply the entire diagram by zero. We denote by Δ_j the categorical dimension of V_j and by $\vartheta(a, b, c)$ the evaluation of the theta graph,

$$\Delta_j = j \bigcirc \quad \vartheta(a, b, c) = \begin{array}{c} a \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ c \end{array} \quad (7.3)$$

Note that $\Delta_j \neq 0$ for all $j \in I$ and $\vartheta(a, b, c) \neq 0$ for all admissible triples $(a, b, c) \in I^3$. For each $j \in I$, we define the vector spaces

$$\omega(V_j) = \text{Hom}(\widehat{V}, \widehat{V} \otimes V_j) = \text{span}_k \left\{ \begin{array}{c} p \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \\ q \quad j \end{array} \quad \middle| \quad p, q \in I \right\} \quad (7.4)$$

and

$$\omega(V_j)^* = \text{Hom}(\widehat{V} \otimes V_j, \widehat{V}) = \text{span}_k \left\{ \begin{array}{c} p \quad j \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \\ q \end{array} \quad \middle| \quad p, q \in I \right\} \quad (7.5)$$

where $\widehat{V} = \bigoplus_{j \in I} V_j$ denotes the *universal object*. This notation is compatible with the remainder of the present article, but *not* with [40]. There, the universal object is called ω whereas our ω is the long forgetful functor. In the following, we prefer the bases $(e_{pq}^{(V_j)})_{pq}$ and $(e_{(V_j)}^{pq})_{pq}$ with

$$e_{pq}^{(V_j)} = \begin{array}{c} p \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \\ q \quad j \end{array} \quad \text{and} \quad e_{(V_j)}^{pq} = \frac{\Delta_q}{\vartheta(p, q, j)} \begin{array}{c} p \quad j \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \\ q \end{array} \quad (7.6)$$

where $p, q \in I$ such that (p, q, j) is admissible. The reconstructed WHA is the vector space

$$H = \bigoplus_{j \in I} \omega(V_j)^* \otimes \omega(V_j). \quad (7.7)$$

A convenient basis of H is given by the vectors of the form

$$[e_{(V_j)}^{pq} | e_{rs}^{(V_j)}]_{V_j} := e_{(V_j)}^{pq} \otimes e_{rs}^{(V_j)} \quad (7.8)$$

for $j \in I$ and $p, q, r, s \in I$ such that (p, q, j) and (r, s, j) are admissible. We can now give the coalgebra structure (H, Δ, ε) :

$$\Delta([e_{(V_j)}^{pq}|e_{rs}^{(V_j)}]_{V_j}) = \sum_{t,u \in I} [e_{(V_j)}^{pq}|e_{tu}^{(V_j)}]_{V_j} \otimes [e_{(V_j)}^{tu}|e_{rs}^{(V_j)}]_{V_j}, \quad (7.9)$$

$$\varepsilon([e_{(V_j)}^{pq}|e_{rs}^{(V_j)}]_{V_j}) = \delta_{ps}\delta_{qr}. \quad (7.10)$$

More generally, the counit is the evaluation of the following trace,

$$\varepsilon\left(\left[\begin{array}{c|c} \boxed{A} & \boxed{B} \end{array} \right] \right) = \begin{array}{c} \text{Diagram: A box labeled } B \text{ on top of a box labeled } A, with a loop labeled } D \text{ on the right side.} \end{array} \quad (7.11)$$

By this notation, we mean that one takes whatever ribbon tangles A and B occur in the argument of ε and pastes them into the diagram on the right. All open ends of the tangles are labeled by simple objects, and when one connects two of them, say labeled by $p \in I$ and $q \in I$, the composition of morphisms is zero unless $p = q$, i.e. one has to write down a prefactor of δ_{pq} . For example, putting

$$\begin{array}{c} r \quad j \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ s \end{array} \quad \text{below} \quad \begin{array}{c} p \\ \bullet \\ \diagup \quad \diagdown \\ q \quad k \end{array} \quad \text{gives} \quad \delta_{qr}\delta_{kj} \quad \begin{array}{c} p \\ \bullet \\ \diagup \quad \diagdown \\ q \quad k \\ \bullet \\ s \end{array} \quad (7.12)$$

The morphism D is defined as

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \bullet \end{array} \quad = \sum_{j \in I} \Delta_j^{-1} \left| \begin{array}{c} j \end{array} \right. \quad (7.13)$$

We extend our notation for the elements of H to $\omega(X) = \text{Hom}(\widehat{V}, \widehat{V} \otimes X)$ and $\omega(X)^* = \text{Hom}(\widehat{V} \otimes X, \widehat{V})$ for any object X of \mathcal{C} . Such elements are denoted by

$$\left[\begin{array}{c|c} \boxed{A} & \boxed{B} \end{array} \right]_X \quad (7.14)$$

and they are indeed elements of H if we impose for each morphism $f: X \rightarrow Y$ the relations

$$\left[\begin{array}{c|c} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} & \boxed{B} \end{array} \right]_X = \left[\begin{array}{c|c} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} & \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \end{array} \right]_Y \quad (7.15)$$

Before we present the algebra structure of H , we recall the recoupling identity

$$\begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ a \quad d \end{array} = \sum_{i \in I} \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}_q \begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ a \quad d \end{array} \quad (7.16)$$

which holds whenever the triples (a, b, j) and (c, d, j) on the left hand side are admissible. Here the *quantum 6j-symbols* can be computed as follows.

$$\left\{ \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\}_q = \frac{\Delta_i}{\vartheta(a, d, i)\vartheta(b, c, i)} \quad \begin{array}{c} \text{Diagram: A diamond shape with vertices } a, b, c, d \text{ and } j \text{ in the center. } a \text{ is bottom-left, } b \text{ is top-left, } c \text{ is top-right, } d \text{ is bottom-right. } i \text{ is to the right of } d. \end{array} \quad (7.17)$$

The algebra structure (H, μ, η) is given by

$$\eta(1) = \sum_{j, \ell \in I} [e_{(V_0)}^{jj} | e_{\ell\ell}^{(V_0)}]_{V_0}, \quad (7.18)$$

$$\mu\left(\left[\begin{array}{c|c} \boxed{A} & \boxed{B} \\ \hline \boxed{} & \boxed{} \end{array} \right]_X \otimes \left[\begin{array}{c|c} \boxed{C} & \boxed{D} \\ \hline \boxed{} & \boxed{} \end{array} \right]_Y\right) = \left[\begin{array}{c|c} \boxed{A} & \boxed{D} \\ \hline \boxed{C} & \boxed{B} \end{array} \right]_{X \otimes Y} \quad (7.19)$$

In terms of our favourite bases, the multiplication reads

$$\begin{aligned} & \mu([e_{(V_j)}^{pq} | e_{rs}^{(V_j)}]_{V_j} \otimes [e_{(V_\ell)}^{ab} | e_{cd}^{(V_\ell)}]_{V_\ell}) \\ &= \delta_{qa} \delta_{rd} \sum_{u \in I} [e_{(V_u)}^{pb} | e_{cs}^{(V_u)}]_{V_u} \left\{ \begin{matrix} p & j & u \\ \ell & b & a \end{matrix} \right\}_q \left\{ \begin{matrix} c & \ell & u \\ j & s & d \end{matrix} \right\}_q \frac{\Delta_a \vartheta(p, b, u) \vartheta(j, \ell, u)}{\Delta_u \vartheta(p, a, j) \vartheta(a, b, \ell)}. \end{aligned} \quad (7.20)$$

The antipode of H is given by

$$S\left(\left[\begin{array}{c|c} \boxed{A} & \boxed{B} \\ \hline \boxed{} & \boxed{} \end{array} \right]_X\right) = \left[\begin{array}{c|c} \boxed{D} & \boxed{B} \\ \hline \boxed{B^{-1}} & \boxed{A} \end{array} \right]_X \quad (7.21)$$

which reads in our basis

$$S([e_{(V_j)}^{pq} | e_{rs}^{(V_j)}]_{V_j}) = [e_{(V_j)}^{rs} | e_{pq}^{(V_j)}]_{V_j} \frac{\Delta_q \vartheta(r, s, j)}{\Delta_r \vartheta(p, q, j)}. \quad (7.22)$$

We finally list the coquasitriangular structure

$$r\left(\left[\begin{array}{c|c} \boxed{A} & \boxed{B} \\ \hline \boxed{} & \boxed{} \end{array} \right]_X \otimes \left[\begin{array}{c|c} \boxed{C} & \boxed{D} \\ \hline \boxed{} & \boxed{} \end{array} \right]_Y\right) = \begin{array}{c} \text{Diagram: A vertical stack of boxes } B, D, A, C \text{ connected by lines. A circle labeled } D \text{ is to the right, connected to the lines between } B \text{ and } D, \text{ and } A \text{ and } C. \end{array} \quad (7.23)$$

and the universal ribbon form

$$\nu\left(\left[\begin{array}{c|c} \boxed{A} & \boxed{B} \\ \hline \boxed{} & \boxed{} \end{array} \right]_X\right) = \begin{array}{c} \text{Diagram: A vertical stack of boxes } B, A \text{ connected by lines. A circle labeled } D \text{ is to the right, connected to the lines between } B \text{ and } A. \end{array} \quad (7.24)$$

A Background on tensor categories

In this appendix, we collect the relevant definitions and properties of monoidal, autonomous, braided monoidal, ribbon and abelian categories, following Freyd–Yetter [41], Schauenburg [13], Turaev [2] and MacLane [39]. In order to keep the appendix short, we write down identities involving morphisms rather than the more familiar commutative diagrams.

A.1 Monoidal categories

Definition A.1. A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a category \mathcal{C} with a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (*tensor product*), an object $\mathbb{1} \in |\mathcal{C}|$ (*monoidal unit*) and natural isomorphisms $\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ (*associator*), $\lambda_X: \mathbb{1} \otimes X \rightarrow X$ (*left-unit constraint*) and $\rho_X: X \otimes \mathbb{1} \rightarrow X$ (*right-unit constraint*) for all $X, Y, Z \in |\mathcal{C}|$, subject to the pentagon axiom

$$\alpha_{X,Y,Z \otimes W} \circ \alpha_{X \otimes Y, Z, W} = (\text{id}_X \otimes \alpha_{Y,Z,W}) \circ \alpha_{X,Y \otimes Z, W} \circ (\alpha_{X,Y,Z} \otimes \text{id}_W) \quad (\text{A.1})$$

and the triangle axiom

$$\rho_X \otimes \text{id}_Y = (\text{id}_X \otimes \lambda_Y) \circ \alpha_{X,\mathbb{1},Y} \quad (\text{A.2})$$

for all $X, Y, Z, W \in |\mathcal{C}|$.

Definition A.2. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes', \mathbb{1}', \alpha', \lambda', \rho')$ be monoidal categories.

- (1) A *lax monoidal functor* $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ consists of a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$, morphisms $F_{X,Y}: FX \otimes' FY \rightarrow F(X \otimes Y)$ that are natural in $X, Y \in |\mathcal{C}|$, and of a morphism $F_0: \mathbb{1}' \rightarrow F\mathbb{1}$, subject to the hexagon axiom

$$F_{X,Y \otimes Z} \circ (\text{id}_{FX} \otimes' F_{Y,Z}) \circ \alpha'_{FX,FY,FZ} = F\alpha_{X,Y,Z} \circ F_{X \otimes Y, Z} \circ (F_{X,Y} \otimes' \text{id}_{FZ}) \quad (\text{A.3})$$

and the two squares

$$\lambda'_{FX} = F\lambda_X \circ F_{\mathbb{1},X} \circ (F_0 \otimes' \text{id}_{FX}), \quad (\text{A.4})$$

$$\rho'_{FX} = F\rho_X \circ F_{X,\mathbb{1}} \circ (\text{id}_{FX} \otimes' F_0) \quad (\text{A.5})$$

for all $X, Y, Z \in |\mathcal{C}|$.

- (2) An *oplax monoidal functor* $(F, F^{X,Y}, F^0): \mathcal{C} \rightarrow \mathcal{C}'$ consists of a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$, morphisms $F^{X,Y}: F(X \otimes Y) \rightarrow FX \otimes' FY$ that are natural in $X, Y \in |\mathcal{C}|$, and of a morphism $F^0: F\mathbb{1} \rightarrow \mathbb{1}'$, subject to the hexagon axiom

$$(\text{id}_{FX} \otimes' F^{Y,Z}) \circ F^{X,Y \otimes Z} \circ F\alpha_{X,Y,Z} = \alpha'_{FX,FY,FZ} \circ (F^{X,Y} \otimes' \text{id}_{FZ}) \circ F^{X \otimes Y, Z} \quad (\text{A.6})$$

and the two squares

$$F\lambda_X = \lambda'_{FX} \circ (F^0 \otimes' \text{id}_{FX}) \circ F^{\mathbb{1},X}, \quad (\text{A.7})$$

$$F\rho_X = \rho'_{FX} \circ (\text{id}_{FX} \otimes' F^0) \circ F^{X,\mathbb{1}} \quad (\text{A.8})$$

for all $X, Y, Z \in |\mathcal{C}|$.

- (3) A *strong monoidal functor* $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ is a lax monoidal functor such that all $F_{X,Y}$, $X, Y \in |\mathcal{C}|$ and F_0 are isomorphisms.

Definition A.3. Let $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ and $(G, G_{X,Y}, G_0): \mathcal{C} \rightarrow \mathcal{C}'$ be lax monoidal functors between monoidal categories \mathcal{C} and \mathcal{C}' . A *monoidal natural transformation* $\eta: F \Rightarrow G$ is a natural transformation such that

$$\eta_{X \otimes Y} \circ F_{X,Y} = G_{X,Y} \circ (\eta_X \otimes' \eta_Y) \quad (\text{A.9})$$

for all $X, Y \in \mathcal{C}$.

There is a similar notion of monoidal natural transformation if the functors are oplax rather than lax. Compositions of [lax, oplax, strong] monoidal functors are again [lax, oplax, strong] monoidal. The following result is well known, but quite laborious to verify.

Proposition A.4. Let \mathcal{C} and \mathcal{C}' be monoidal categories and $F \dashv G: \mathcal{C}' \rightarrow \mathcal{C}$ be an adjunction with unit $\eta: 1_{\mathcal{C}} \Rightarrow G \circ F$ and counit $\varepsilon: F \circ G \Rightarrow 1_{\mathcal{C}'}$.

- (1) If F has an oplax monoidal structure (F, F^{C_1, C_2}, F^0) , then G has a lax monoidal structure (G, G_{D_1, D_2}, G_0) as follows,

$$G_{D_1, D_2} = G(\varepsilon_{D_1} \otimes \varepsilon_{D_2}) \circ G(F^{G(D_1), G(D_2)}) \circ \eta_{G(D_1) \otimes G(D_2)}, \quad (\text{A.10})$$

$$G_0 = G(F^0) \circ \eta_1. \quad (\text{A.11})$$

- (2) If F is strong monoidal, then both η and ε are monoidal natural transformations.
 (3) If F is strong monoidal and the adjunction is an equivalence, then G is strong monoidal.

By an equivalence of monoidal categories, we mean an equivalence of categories such that one of the functors is strong monoidal. One can then chose the other functor in such a way that one has an adjoint equivalence and apply Proposition A.4, items (2) and (3). We denote such an equivalence by $\mathcal{C} \simeq_{\otimes} \mathcal{D}$.

A.2 Duality

Definition A.5. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category.

- (1) A *left-dual* $(X^*, \text{ev}_X, \text{coev}_X)$ of an object $X \in |\mathcal{C}|$ consists of an object $X^* \in |\mathcal{C}|$ and morphisms $\text{ev}_X: X^* \otimes X \rightarrow \mathbb{1}$ (*left evaluation*) and $\text{coev}_X: \mathbb{1} \rightarrow X \otimes X^*$ (*left coevaluation*) that satisfy the triangle identities

$$\rho_X \circ (\text{id}_X \otimes \text{ev}_X) \circ \alpha_{X, X^*, X} \circ (\text{coev}_X \otimes \text{id}_X) \circ \lambda_X^{-1} = \text{id}_X, \quad (\text{A.12})$$

$$\lambda_{X^*} \circ (\text{ev}_X \otimes \text{id}_{X^*}) \circ \alpha_{X^*, X, X^*}^{-1} \circ (\text{id}_{X^*} \otimes \text{coev}_X) \circ \rho_{X^*}^{-1} = \text{id}_{X^*}. \quad (\text{A.13})$$

- (2) A *right-dual* $(\overline{X}, \overline{\text{ev}}_X, \overline{\text{coev}}_X)$ of $X \in |\mathcal{C}|$ consists of an object $\overline{X} \in |\mathcal{C}|$ and morphisms $\overline{\text{ev}}_X: X \otimes \overline{X} \rightarrow \mathbb{1}$ (*right evaluation*) and $\overline{\text{coev}}_X: \mathbb{1} \rightarrow \overline{X} \otimes X$ (*right coevaluation*) that satisfy the triangle identities

$$\lambda_X \circ (\overline{\text{ev}}_X \otimes \text{id}_X) \circ \alpha_{X, \overline{X}, X}^{-1} \circ (\text{id}_X \otimes \overline{\text{coev}}_X) \circ \rho_X^{-1} = \text{id}_X, \quad (\text{A.14})$$

$$\rho_{\overline{X}} \circ (\text{id}_{\overline{X}} \otimes \overline{\text{ev}}_X) \circ \alpha_{\overline{X}, X, \overline{X}} \circ (\overline{\text{coev}}_X \otimes \text{id}_{\overline{X}}) \circ \lambda_{\overline{X}}^{-1} = \text{id}_{\overline{X}}. \quad (\text{A.15})$$

Definition A.6. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category and $f: X \rightarrow Y$ be a morphism of \mathcal{C} .

(1) If both X and Y have left-duals, the *left-dual* of f is defined as

$$f^* := \lambda_{X^*} \circ (\text{ev}_Y \otimes \text{id}_{X^*}) \circ \alpha_{Y^*, Y, X^*}^{-1} \circ (\text{id}_{Y^*} \otimes (f \otimes \text{id}_{X^*})) \circ (\text{id}_{Y^*} \otimes \text{coev}_X) \circ \rho_{Y^*}^{-1}. \quad (\text{A.16})$$

(2) If both X and Y have right-duals, the *right-dual* of f is defined as

$$\bar{f} := \rho_{\bar{X}} \circ (\text{id}_{\bar{X}} \otimes \bar{\text{ev}}_Y) \circ \alpha_{\bar{X}, Y, \bar{Y}} \circ ((\text{id}_{\bar{X}} \otimes f) \otimes \text{id}_{\bar{Y}}) \circ (\bar{\text{coev}}_X \otimes \text{id}_{\bar{Y}}) \circ \lambda_{\bar{Y}}^{-1}. \quad (\text{A.17})$$

Definition A.7. A *[left-, right-]autonomous category* is a monoidal category in which each object is equipped with a specified [left-, right-]dual. An *autonomous category* is a monoidal category that is both left- and right-autonomous.

Note that every autonomous category is monoidally closed because the functor $- \otimes X^*$ is a right adjoint of $- \otimes X$ and $\bar{X} \otimes -$ is a right-adjoint of $X \otimes -$ for all $X \in |\mathcal{C}|$. In particular, the tensor product in an autonomous category preserves colimits in both arguments.

A.3 Ribbon categories

Definition A.8. A *braided monoidal category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \sigma)$ is a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ with natural isomorphisms $\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ for all $X, Y \in |\mathcal{C}|$ that satisfy the two hexagon axioms

$$\sigma_{X \otimes Y, Z} = \alpha_{Z, X, Y} \circ (\sigma_{X, Z} \otimes \text{id}_Y) \circ \alpha_{X, Z, Y}^{-1} \circ (\text{id}_X \otimes \sigma_{Y, Z}) \circ \alpha_{X, Y, Z}, \quad (\text{A.18})$$

$$\sigma_{X, Y \otimes Z} = \alpha_{Y, Z, X}^{-1} \circ (\text{id}_Y \otimes \sigma_{X, Z}) \circ \alpha_{Y, X, Z} \circ (\sigma_{X, Y} \otimes \text{id}_Z) \circ \alpha_{X, Y, Z}^{-1} \quad (\text{A.19})$$

for all $X, Y, Z \in |\mathcal{C}|$. The category is called *symmetric monoidal* if in addition

$$\sigma_{Y, X} \circ \sigma_{X, Y} = \text{id}_{X \otimes Y} \quad (\text{A.20})$$

for all $X, Y \in |\mathcal{C}|$.

Definition A.9. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \sigma)$ and $(\mathcal{C}', \otimes', \mathbb{1}', \alpha', \lambda', \rho', \sigma')$ be braided monoidal categories. A lax monoidal functor $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ is called *braided* if

$$F \sigma_{X,Y} \circ F_{X,Y} = F_{Y,X} \circ \sigma'_{F_X, F_Y} \quad (\text{A.21})$$

for all $X, Y \in |\mathcal{C}|$.

Proposition A.10. Let \mathcal{C} and \mathcal{C}' be braided monoidal categories and $F \dashv G: \mathcal{C}' \rightarrow \mathcal{C}$ be an adjoint equivalence. If F is strong monoidal and braided, then so is G .

By an equivalence of braided monoidal categories, we therefore mean an equivalence of categories one functor of which is strong monoidal and braided.

Definition A.11. A *ribbon category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$ is a left-autonomous category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev})$ that is braided monoidal as $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \sigma)$ with natural isomorphisms (*ribbon twist*) $\nu_X: X \rightarrow X$ such that

$$\nu_{X \otimes Y} = \sigma_{Y, X} \circ \sigma_{X, Y} \circ (\nu_X \otimes \nu_Y) \quad (\text{A.22})$$

and

$$(\nu_X \otimes \text{id}_{X^*}) \circ \text{coev}_X = (\text{id}_X \otimes \nu_{X^*}) \circ \text{coev}_X \quad (\text{A.23})$$

for all $X, Y \in |\mathcal{C}|$.

Note that in every ribbon category \mathcal{C} , there are natural isomorphisms $\tau_X: X \rightarrow X^{**}$ for all $X \in |\mathcal{C}|$, given by

$$\tau_X = \lambda_{X^{**}} \circ (\text{ev}_X \otimes \text{id}_{X^{**}}) \circ (\sigma_{X, X^*} \otimes \text{id}_{X^{**}}) \circ (\nu_X \otimes \text{coev}_{X^*}) \circ \rho_X^{-1}, \quad (\text{A.24})$$

that satisfy $(\tau_X)^* = \tau_{X^*}^{-1}$.

Every ribbon category \mathcal{C} is not only left-autonomous, but also right-autonomous with $(\overline{X}, \overline{\text{ev}}_X, \overline{\text{coev}}_X)$ where $\overline{X} = X^*$ and

$$\overline{\text{ev}}_X = \text{ev}_X \circ \sigma_{X, X^*} \circ (\nu_X \otimes \text{id}_{X^*}), \quad (\text{A.25})$$

$$\overline{\text{coev}}_X = (\text{id}_{X^*} \otimes \nu_X) \circ \sigma_{X, X^*} \circ \text{coev}_X \quad (\text{A.26})$$

for all $X \in |\mathcal{C}|$. The left- and the right-dual of any morphism $f: X \rightarrow Y$ agree, $f^* = \overline{f}$.

Definition A.12. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$ be a ribbon category, $X \in |\mathcal{C}|$, and $f: X \rightarrow X$ be a morphism of \mathcal{C} . Then we define

(1) the *trace* of f by

$$\text{tr}_X(f) := \overline{\text{ev}}_X \circ (f \otimes \text{id}_{X^*}) \circ \text{coev}_X: \mathbb{1} \rightarrow \mathbb{1}, \quad (\text{A.27})$$

(2) the *dimension* of X by

$$\dim(X) := \text{tr}_X(\text{id}_X). \quad (\text{A.28})$$

Proposition A.13. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$ be a ribbon category. Then

- (1) $\text{tr}_X(f) = \text{tr}_{X^*}(f^*)$ for all $f: X \rightarrow X$.
- (2) $\text{tr}_X(g \circ f) = \text{tr}_Y(f \circ g)$ for all $f: X \rightarrow Y$ and $g: Y \rightarrow X$.
- (3) $\text{tr}_{X_1 \otimes X_2}(h_1 \otimes h_2) = \text{tr}_{X_1}(h_1) \text{tr}_{X_2}(h_2)$ for all $h_j: X_j \rightarrow X_j$, $j \in \{1, 2\}$.

Definition A.14. Let \mathcal{C} and \mathcal{C}' be ribbon categories. A *ribbon functor* $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ is a lax monoidal functor that is braided and satisfies

$$F\nu_X = \nu'_{FX} \quad (\text{A.29})$$

for all $X \in |\mathcal{C}|$.

Proposition A.15. Let \mathcal{C} and \mathcal{C}' be ribbon categories and $F \dashv G: \mathcal{C}' \rightarrow \mathcal{C}$ be an adjoint equivalence. If F is strong monoidal and ribbon, then so is G .

By an equivalence of ribbon categories we therefore mean an equivalence of categories one functor of which is strong monoidal and ribbon. The following proposition states what strong monoidal ribbon functors do to traces.

Proposition A.16. Let \mathcal{C} and \mathcal{C}' be ribbon categories and $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ be a strong monoidal ribbon functor. Then for each morphism $f: X \rightarrow X$ of \mathcal{C} , the diagram

$$\begin{array}{ccc} \mathbb{1}' & \xrightarrow{F_0} & F\mathbb{1} \\ \text{tr}_{FX}(Ff) \downarrow & & \downarrow F\text{tr}_X(f) \\ \mathbb{1}' & \xrightarrow{F_0} & F\mathbb{1} \end{array} \quad (\text{A.30})$$

commutes.

A.4 Abelian and semisimple categories

Definition A.17. A category \mathcal{C} is called **Ab-enriched** if it is enriched in the category **Ab** of abelian groups, i.e. if $\text{Hom}(X, Y)$ is an abelian group for all objects $X, Y \in |\mathcal{C}|$ and if the composition of morphisms is \mathbb{Z} -bilinear.

Let k be a commutative ring. A category \mathcal{C} is called k -linear if it is enriched in ${}_k\mathcal{M}$, the category of k -modules, i.e. if $\text{Hom}(X, Y)$ is a k -module for all $X, Y \in |\mathcal{C}|$ and if the composition of morphisms is k -bilinear.

A functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ between [**Ab-enriched**, k -linear] categories is called [additive, k -linear] if it induces homomorphisms of [additive groups, k -modules]

$$\text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY) \quad (\text{A.31})$$

for all $X, Y \in |\mathcal{C}|$.

Definition A.18. A monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is called [**Ab-enriched**, k -linear] if \mathcal{C} is [**Ab-enriched**, k -linear] and if the tensor product of morphisms is [\mathbb{Z} -bilinear, k -bilinear].

Definition A.19. An *additive category* is an **Ab-enriched** category that has a terminal object and all binary products. A *preabelian category* is an **Ab-enriched** category that has all finite limits. An *abelian category* is a preabelian category in which every monomorphism is a kernel and in which every epimorphism is a cokernel.

A functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ between preabelian categories is called *exact* if it preserves all finite limits.

Recall that in an **Ab-enriched** category, an object is terminal if and only if it is initial and if and only if it is null. Every additive category has all finite biproducts. An equivalence of [**Ab-enriched**, k -linear] categories is an equivalence of categories one functor of which is [additive, k -linear].

Definition A.20. Let \mathcal{C} be a k -linear category, k a commutative ring.

- (1) An object $X \in |\mathcal{C}|$ is called *simple* if $\text{End}(X) \cong k$ are isomorphic as k -modules.
- (2) An object $X \in |\mathcal{C}|$ is called *null* if $\text{End}(X) \cong \{0\}$.
- (3) The category \mathcal{C} is called *semisimple* if there exists a family $\{V_j\}_{j \in I}$ of objects $V_j \in |\mathcal{C}|$, I some index set, such that
 - (a) V_j is simple for all $j \in I$.
 - (b) $\text{Hom}(V_j, V_\ell) = \{0\}$ for all $j, \ell \in I$ for which $j \neq \ell$.
 - (c) For each object $X \in |\mathcal{C}|$, there is a finite sequence $j_1^{(X)}, \dots, j_{n^X}^{(X)} \in I$, $n^X \in \mathbb{N}_0$, and morphisms $\iota_\ell^{(X)}: V_{j_\ell} \rightarrow X$ and $\pi_\ell^{(X)}: X \rightarrow V_{j_\ell}$ such that

$$\text{id}_X = \sum_{\ell=1}^{n^X} \iota_\ell^X \circ \pi_\ell^X. \quad (\text{A.32})$$

and

$$\pi_\ell^X \circ \iota_m^X = \begin{cases} \text{id}_{V_{j_\ell^X}}, & \text{if } \ell = m, \\ 0, & \text{else} \end{cases} \quad (\text{A.33})$$

- (4) The category is called *finitely semisimple* (also *Artinian semisimple*) if it is semisimple with a finite index set I in condition (3).

Proposition A.21 (see [2, Lemma II.4.2.2]). Let \mathcal{C} be a k -linear category, k a commutative ring. If \mathcal{C} is [finitely] semisimple, then there is a [finite] set $\mathcal{J} \subseteq |\mathcal{C}|$ of objects each of which is non-null such that

$$\begin{aligned} \Phi: \bigoplus_{J \in \mathcal{J}} \text{Hom}(X, J) \otimes \text{Hom}(J, Y) &\rightarrow \text{Hom}(X, Y), \\ f \otimes g &\mapsto g \circ f, \end{aligned} \tag{A.34}$$

is an isomorphism for all $X, Y \in |\mathcal{C}|$.

Lemma A.22. Let \mathcal{C} be a k -linear category, k a field, and $\text{Hom}(X, Y)$ be a finite-dimensional vector space over k for all $X, Y \in |\mathcal{C}|$ and let \mathcal{J} be a set of objects that satisfies the conditions of Proposition A.21.

- (1) Each $J \in \mathcal{J}$ is simple.
- (2) If $X \in |\mathcal{C}|$ is simple, then there exists some $J_X \in \mathcal{J}$ such that $X \cong J_X$. For all other $J \in \mathcal{J}$, $J \not\cong J_X$, we have $\text{Hom}(X, J) = \{0\} = \text{Hom}(J, X)$.
- (3) If $X, Y \in |\mathcal{C}|$ are both simple, then either $X \cong Y$ or $\text{Hom}(X, Y) = \{0\}$.

Proof. The idea for this proof is that both source and target of the isomorphism (A.34) are finite-dimensional vector spaces over k . We can therefore count dimensions. \square

Corollary A.23. Let \mathcal{C} be a semisimple k -linear category with family $\{V_j\}_{j \in I}$ of simple objects, k a field, and $\text{Hom}(X, Y)$ be a finite-dimensional vector space over k for all $X, Y \in |\mathcal{C}|$. If $X \in \mathcal{C}$ is simple, then there exists some $j \in I$ such that $X \cong V_j$.

A.5 Ab-enriched and non-degenerate ribbon categories

Proposition A.24. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$ be an **Ab**-enriched ribbon category.

- (1) The abelian group $k := \text{End}(\mathbb{1})$ is a unital commutative ring with respect to the composition of morphisms.
- (2) The category \mathcal{C} is k -linear as a monoidal category.
- (3) For all objects $X \in |\mathcal{C}|$, the trace

$$\text{tr}_X: \text{Hom}(X, X) \rightarrow k, \tag{A.35}$$

is k -linear.

- (4) For all objects $X, Y \in |\mathcal{C}|$, the map

$$\text{Hom}(Y, X) \otimes \text{Hom}(X, Y) \rightarrow k, \quad f \otimes g \mapsto \text{tr}_X(f \circ g) \tag{A.36}$$

is k -bilinear.

Definition A.25. An **Ab**-enriched ribbon category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$ is called *non-degenerate* if the bilinear forms (A.36) are non-degenerate for all objects $X, Y \in |\mathcal{C}|$, i.e. if $\text{tr}_X(f \circ g) = 0$ for all $g: X \rightarrow Y$ implies $f = 0$.

If we work with semisimple ribbon categories, we also require the set of representatives of the simple objects to contain the monoidal unit and to be closed under duality.

Definition A.26. An **Ab**-enriched ribbon category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \sigma, \nu)$, $k = \text{End}(\mathbb{1})$, is called [finitely] *semisimple* if the underlying k -linear category is [finitely] semisimple and the family $\{V_j\}_j$ of Definition A.20(3) satisfies the following conditions.

- (1) There is an element $0 \in I$ such that $V_0 \cong \mathbb{1}$.
- (2) For each $j \in I$, there is some $j^* \in I$ such that $V_{j^*} \cong V_j^*$.

Proposition A.27. Let \mathcal{C} be a semisimple **Ab**-enriched ribbon category with family $\{V_j\}_{j \in I}$ as in Definition A.20(3). Then for all $j \in I$, $\dim V_j$ is invertible in k [2, Lemma II.4.2.4].

The following proposition gives conditions under which ribbon functors preserve traces.

Proposition A.28. Let \mathcal{C} and \mathcal{C}' be semisimple k -linear ribbon categories, k a field, and $(F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'$ be a strong monoidal k -linear ribbon functor. Then for each morphism $f: X \rightarrow X$ of \mathcal{C} ,

$$\text{tr}_X(f) = \text{tr}_{F_X}(Ff). \quad (\text{A.37})$$

Proof. Since the monoidal units of \mathcal{C} and \mathcal{C}' are simple, $\text{tr}_X(f) = \lambda_f \text{id}_{\mathbb{1}}$ and $\text{tr}_{Ff}(Ff) = \lambda'_f \text{id}_{F\mathbb{1}}$, for some $\lambda_f, \lambda'_f \in k$. By k -linearity of F , $F \text{tr}_X(f) = \lambda_f \text{id}_{F\mathbb{1}}$, and so (A.30) implies that $\lambda_f \text{id}_{F\mathbb{1}} = \lambda'_f \text{id}_{F\mathbb{1}}$ and therefore $\lambda_f = \lambda'_f$. \square

Proposition A.29. Let \mathcal{C} be an **Ab**-enriched non-degenerate ribbon category, $k = \text{End}(\mathbb{1})$ be a field and $\text{Hom}(X, Y)$ be a finite-dimensional vector space over k for all $X, Y \in |\mathcal{C}|$. If \mathcal{C} satisfies all conditions of a finitely semisimple category of Definition A.20 except maybe for (A.33), then the $\iota_\ell^{(X)}$ and $\pi_\ell^{(X)}$ can be chosen in such a way that (A.33) holds as well.

Proof. Consider the bilinear form

$$\begin{aligned} \Psi: \text{Hom}(X, \widehat{V}) \otimes \text{Hom}(\widehat{V}, X) &\rightarrow k, \\ f \otimes g &\mapsto \sum_{\ell=1}^{n(X)} \text{tr}_{\widehat{V}}(f \circ \iota_\ell^{(X)} \circ \pi_\ell^{(X)} \circ g) (\dim V_{j_\ell^{(X)}})^{-1} \end{aligned} \quad (\text{A.38})$$

which is non-degenerate by Proposition A.27. The $(\iota_\ell^{(X)})_\ell$ form a basis of $\text{Hom}(\widehat{V}, X)$, and since k is a field and the Hom spaces are finite-dimensional vector spaces, we can choose a dual basis $(\tilde{\pi}_\ell^{(X)})_\ell$ of $\text{Hom}(X, \widehat{V})$. Then for any $1 \leq p, q \leq n(X)$ and $p \neq q$, $0 = \Psi(\tilde{\pi}_p^{(X)} \otimes \iota_q^{(X)})$ implies $\tilde{\pi}_p^{(X)} \circ \iota_q^{(X)} = 0$ by non-degeneracy. Finally, if $p = q$, $1 = \Psi(\tilde{\pi}_p^{(X)} \otimes \iota_p^{(X)})$ implies that $\tilde{\pi}_p^{(X)} \circ \iota_p^{(X)} = \text{id}_{V_{j_p^{(X)}}}$ because $V_{j_p^{(X)}}$ is simple. \square

Acknowledgements

The author would like to thank Gabriella Böhm, Shahn Majid and Kornél Szlachányi for valuable discussions and everyone at RMKI Budapest for their hospitality.

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